GUIDE TO THE STUDY OF MATHEMATICS.

Stewart's Mathematical Series.

GUIDE TO THE STUDY OF MATHEMATICS

For the B.A. and B.Sc. Examinations of the University of London.

BY

ARTHUR L. SPARKES, B.A., F.C.S.,

AUTHOR OF 'ALGEBRAIC FACTORS SIMPLIFIED,' 'GUIDE TO THE MATHEMATICS OF THE MATRICULATION EXAMINATION,' ETC.;

HEAD MASTER OF THE MASONIC INSTITUTION FOR BOYS FOR IRELAND.

PART I.

ARITHMETIC.
ALGEBRA.
GEOMETRY.
TRIGONOMETRY.

LONDON:
W. STEWART & CO., HOLBORN VIADUCT STEPS, E.C.
EDINBURGH & GLASGOW: MENZIES & CO.
1879.

GUIDE TO THE STUDY OF MATHEMATICS.

INTRODUCTION

THE following pages are intended as a guide to the Mathematical Course for the London Examinations only. Information which is easily obtained in ordinary text-books is not given, but advice is offered as to the best book to refer to. Subjects which are not usually given, or points where the student would meet with difficulties, are more fully explained, as in the chapter on Mensuration.

The Second Part will contain all the subjects in Coordinate Geometry which are required for these examinations in full, with simple exercises from examination papers which have been already given.

DUBLIN, *Sept.* 1879.

GUIDE TO THE STUDY OF MATHEMATICS.

The 1st B.A. and 1st B.Sc. Examinations.

TWO papers are given in Mathematics at each of these examinations. The first paper embraces Arithmetic and Algebra, the second Geometry, Trigonometry, and Conics. The syllabus is divided into four sections:—(1) Arithmetic; (2) Algebra; (3) Geometry, including Conics; (4) Trigonometry.

I. ARITHMETIC.

It will be convenient for us to take the sections of the syllabus in the order given, beginning with *Arithmetic.*

The subjects included under this head are:—

The ordinary rules of arithmetic.
Vulgar and decimal fractions.
Rule of three and its applications.
Simple and compound interest.
Present value, discount, and annuities.
Extraction of square root.
Nature and use of logarithms.

(*a*) THE ORDINARY RULES OF ARITHMETIC.

We recommend Barnard Smith's *Arithmetic* as the most useful; Colenso's is also a good one. Examples given in rules not mentioned in the sections following are seldom given; if ever they are, examples in practice, stocks, and factors of numbers are generally selected.

Care should be taken to get up the application of Practice. For example, suppose we [illegible] to know

what a bankrupt will pay on a debt of £1250, 10s., if his assets realize 3s. 4d. in the pound:—

	£1250	10	0
3s. 4d. = $\frac{1}{6}$.	£208	8	4

Or the sum of 1000 guineas, £1000, 1000 sixpences, 1000 pence, and 1000 farthings. Taking one coin of each sort and adding their values together, we get:—

£1	1	0	for a guinea.
1	0	0	„ sovereign.
		6	„ sixpence.
		1	„ penny.
		$\frac{1}{4}$	„ farthing.
Total, £2	1	$7\frac{1}{4}$	

Now, 1000 of each of these coins will amount to 1000 at £2, 1s. $7\frac{1}{4}$d. each.

Stocks are thoroughly explained, and a very complete set of examples given, in each of the books we have recommended.

In 1870 the following question was given:—'A person invested in the three per cents. at $94\frac{1}{2}$, and received as interest £200 a year. What sum did he invest?'

This may be done by rule of three, or thus:—

£3 is realized by investing £$94\frac{1}{2}$;

$$\therefore\ 1 \quad „ \quad „ \quad „ \quad \frac{94\frac{1}{2}}{3};$$

$$\therefore\ 200 \quad „ \quad „ \quad „ \quad \frac{94\frac{1}{2}\times 200}{3};$$

$$\text{or } \frac{189\times 200}{2\times 3}=\frac{189\times 100}{3}=6300.$$

A knowledge of arithmetical factors is of great importance. Questions involving them occur at almost every examination. It is therefore important to get a thorough knowledge of the rules relating to them. We here enumerate them:—

(1) Every even number is divisible by 2.

(2) If the sum of the digits of a number is a multiple of 3, the number is divisible by 3.

(3) Any number is divisible by 4 if the two right-hand figures are divisible by 4.

(4) Any number is divisible by 5 if the unit figure is 5 or 0.

(5) Any number is divisible by 6 if it is even, and the sum of its digits is a multiple of 3.

(6) Any number is divisible by 8 if the last three figures are divisible by 8.

(7) Any number is divisible by 9 if the sum of its digits is a multiple of 9.

(8) Any number is divisible by 10 if the unit's digit is zero.

(9) Any number is divisible by 11 if the difference of the sums of the alternate digits is zero, or any multiple of 11.

(10) Any number is divisible by 12 if the last two figures are divisible by 4, and the sum of the digits is a multiple of three.

These rules may, of course, be considerably increased, but these are sufficient for all purposes. Of course, if two of these rules apply to a number, the number is divisible by the product of the numbers to which the rules refer. For instance, if rules 3 and 7 apply to a number, it will be divisible by 36.

It will be a valuable exercise for a student to prove these rules. We give a few examples of the proofs. Take rule (6).

It is clear that if the three right-hand figures are divisible by 8, the whole number will be, for all the other figures together represent so many thousands. Now, one thousand is divisible by 8, therefore any number of thousands must be divisible by 8. Therefore, if the three figures on the right are divisible by 8, the whole number must be divisible by 8.

Again, rule (7). Suppose we take a number, say 1234.

$$\begin{aligned}1234 &= 1000 + 200 + 30 + 4\\ &= (999+1)+(198+2)+(27+3)+4\\ &= 999+198+27+1+2+3+4.\end{aligned}$$

But 999, 198, and 27 are divisible by 9. If, therefore, $1+2+3+4$ is divisible by 9, the whole number will be divisible by 9. But $1+2+3+4$ is the sum of the digits of the given number; $\therefore$ if the sum of the digits is divisible by 9, the whole number is divisible by 9.

Rules (7) and (9) may be proved *generally* by algebra. Suppose r to be the radix of a system of notation (that is 10 in the common system), we may show that if the sum of the digits be a multiple of $r-1$ (that is 9 in the

common system), the number will be divisible by $r-1$, or 9.

Let a, b, c, d be the digits of a number. Then the number will be ar^3+br^2+cr+d,

$$\begin{aligned}&\text{or } (ar^3-a+a)+(br^2-b+b)+(cr-c+c)+d\\&=(ar^3-a)+(br^2-b)+(cr-c)+a+b+c+d\\&=a(r^3-1)+b(r^2-1)+c(r-1)+a+b+c+d.\end{aligned}$$

Now, $a(r^3-1)$, $b(r^2-1)$, $c(r-1)$ are all divisible by $r-1$; $\therefore$ if $a+b+c+d$ is divisible by $r-1$, all the quantity is divisible by $r-1$.

But $a+b+c+d$ is the sum of the digits; $\therefore$ if the sum of the digits is divisible by $r-1$, the whole quantity is divisible.

Again, to prove rule (9). Let a, b, c, d be the digits, r the radix. Then, if $(d+b)-(a+c)$ is divisible by $r+1$, the whole quantity will be.

The number will be ar^3+br^2+cr+d, as before; and if this be divided by the ordinary process, we have

$$\begin{array}{l}r+1\,\big)\,ar^3+br^2+cr+d\,\big(\,ar^2+r(b-a)+(c-b+a)\\\quad\underline{ar^3+ar^2}\\\qquad r^2(b-a)\\\qquad \underline{r^2(b-a)+r(b-a)}\\\qquad\qquad r(c-b+a)\\\qquad\qquad \underline{r(c-b+a)+(c-b+a)}\\\qquad\qquad\qquad d-c+b-a\\\qquad\qquad\qquad =(d+b)-(a+c);\end{array}$$

$\therefore$ if $(d+b)-(a+c)$ is divisible by $r+1$, the whole quantity ar^3+br^2+cr+d is divisible.

But $(d+b)-(a+c)$ is the difference of the sums of the digits taken alternately.

Hence if $r=10$, then $r+1=11$.

If 10 be the radix, the number will be divisible by 11, when the difference of the sums formed by adding the alternate digits together is zero, or any multiple of 11.

We now show, by examples already given in these examinations, how these factors may be used:—

Ex. 1. Find the prime factors of 6930, 1470, and 5775, and use them for calculating—(1) The sum of the reciprocals, and (2) The square root of the product of these three numbers (Matric. 1876).

The prime factors of 6930 are 2.5.3.3.7.11 (by rules 1, 4, 2, 9).

The prime factors of 1470 are 2.5.3.7.7 (by rules 1, 4, 2).
The prime factors of 5775 are 5.5.11.3.7.
The sum of the reciprocals will be

$$\frac{1}{6930}+\frac{1}{1470}+\frac{1}{5775};$$

$$i.e.\ \frac{1}{2.5.3.3.7.11}+\frac{1}{2.5.3.7.7}+\frac{1}{5.5.11.3.7}.$$

Now, to bring these to a common denominator, we must find the least common multiple—that is, we must find a number which will contain all the factors of each number, and it must be the smallest number containing them.

Thus, 5 occurs twice in the third denominator; ∴ 5×5 will be a multiple of all the 5's in the three numbers, for 5 only occurs once in each of the other two denominators. Similarly, 7×7 will be the proper multiple of the 7's. Similarly, 3×3 will be the least multiple of the 3's. And as 11 and 2 do not occur more than once in the same denominator, ∴ one 11 and one 2 only will occur in the L. C. M.

Hence the L. C. M. will be 5.5.7.7.2.3.3.11.

The first fraction will be

$$\frac{1\times 5\times 7}{2.5.3.3.7.11\times(5\times 7)}=\frac{35}{2.5.3.3.7.11.5.7}.$$

The second will be

$$\frac{1\times 5\times 3\times 11}{2.5.3.7.7\times(5\times 3\times 11)}=\frac{165}{2.5.3.7.7.5.3.11}.$$

The third will be

$$\frac{1\times 2\times 3\times 7}{5.5.11.3.7\times(2\times 3\times 7)}=\frac{42}{5.5.11.3.7.2.3.7};$$

∴ the sum

$$=\frac{35+165+42}{5.5.7.7.2.3.3.11}=\frac{242}{5.5.7.7.2.3.3.11}.$$

It is clear now that 2 and 11 will divide both numerator and denominator; ∴ the quantity

$$=\frac{11}{5.5.7.7.3.3}=\frac{11}{25\times 49\times 9}=\frac{11}{11025},\ \text{Ans.}$$

Now, to find the square root of the product of the three numbers.

The factors of 6930 are 2.5.3.3.7.11.
" 1470 " 2.5.3.7.7.
" 5775 " 5.5.11.3.7.

Hence the product of all the numbers is

$$(2.5.3.3.7.11)\times(2.5.3.7.7)\times(5.5.11.3.7)$$
$$=2^2.5^2.5^2.3^2.3^2.7^2.7^2.11^2;$$

$\therefore$ the square root is

$$\sqrt{(2\times5\times5\times3\times3\times7\times7\times11)^2}$$
$$=2\times5\times5\times3\times3\times7\times7\times11$$
$$=10\times15\times21\times77$$
$$=150\times21\times77$$
$$=242550, \text{ Ans.}$$

In case more than two numbers are given, of which the greatest common measure is required, it is usually much shorter to use the method of factors.

The general proofs of the rules referred to in this section must be carefully mastered, such as the rule for finding the greatest common measure by the ordinary method.

Ex. 2. If C is a common measure of A and B, show that it is a measure of A + B and A − B.

To prove this, let C be contained in A, n times; $\therefore$ $A=nC$.

Let C be contained in B, m times; $\therefore$ $B=mC$;

$$\therefore A+B=nC+mC=C(n+m);$$

and $A-B=nC-mC=C(n-m)$.

But C is evidently a measure of $C(n+m)$, and also of $C(n-m)$. It is therefore a measure of their equals $A+B$ and $A-B$.

A knowledge of *algebraic* factors is often valuable in working the arithmetical examples. For instance,

Ex. Simplify $\dfrac{\cdot05\times\cdot05\times\cdot05+1}{105}$ (1869).

Now this may be written $\dfrac{(\cdot05)^3+1}{\cdot05+1}$, and thus it is seen that it corresponds in form with $\dfrac{a^3+b^3}{a+b}$, where $a=\cdot05$ and $b=1$.

$$\text{But } \frac{a^3+b^3}{a+b}=a^2-ab+b^2;$$

$$\therefore \frac{(\cdot05)^3+1}{\cdot05+1}=(\cdot05)^2-\cdot05+1$$
$$=\cdot0025-\cdot05+1$$
$$=1\cdot0025-\cdot05$$
$$=\cdot9525.$$

Another example of this kind will be found in the introduction to 'Factors Simplified.' Many examples are frequently given to which it is difficult to apply any special rule, and the examiners properly assume that the candidate has had considerable experience in working examples of this kind. The miscellaneous examples at the end of Barnard Smith's *Arithmetic* should be well got up. Examples relating to the position of the hands of a clock between certain hours are not uncommon, and are sometimes given at the beginning of the paper to indicate that they are expected to be done by arithmetic.

Ex. A person inquiring the time of day is told that it is between 5 and 6 o'clock, and the hands are together. What time is it? (1874.)

The position of this question leads us to believe it is intended to be done by arithmetic.

Thus: the minute hand of a clock goes over 12 minute spaces while the hour hand goes over 1 of them; it therefore gains 11 spaces during the time the hour hand goes over 1.

Now, whatever number of minute spaces the hour hand travels over after 5 o'clock, before the minute hand reaches it, the minute hand will have to travel over the same number $+25$ minute spaces, *i.e.* the minute hand will have to gain 25 minute spaces on the hour hand.

But it gains 11 minute spaces while the hour hand travels over one minute space; $\therefore$ it will gain 25 while the hour hand travels over $\frac{25}{11}$ minute spaces, *i.e.* $2\frac{3}{11}$ minute spaces; $\therefore$ they will be together at $25+2\frac{3}{11}$ or $27\frac{3}{11}$ minutes past 5.

Sometimes examples like the following are given:—

Which is the greater, $\sqrt{2}$ or $\sqrt[3]{3}$? In cases of this kind, no roots should be extracted, for the one which is greater will produce the greater number when they are both raised to a power which will clear all the roots away; for instance, if each of these be raised to the sixth power,

$$(\sqrt{2})^6 = 2^3 = 8,$$
$$(\sqrt[3]{3})^6 = 3^2 = 9.$$

Now 9 is greater than 8; $\therefore$ $\sqrt[3]{3}$ is greater than $\sqrt{2}$.

(*b*) Vulgar and Decimal Fractions.

A student who has passed the matriculation will have but little difficulty with the subjects mentioned in this section. The examples in fractions usually given at the matriculation show that a thorough knowledge of them is expected at that examination. When questions are given in fractions, whether vulgar or decimal, they usually relate to rules. It is therefore important that the rules respecting the position of the point, the reduction of circulating decimals to vulgar fractions, etc., should be well understood.

In reducing vulgar fractions to their lowest terms, it is often neater to take out the factors common to both numerator and denominator one by one, than to find the greatest common measure.

Every fraction, when reduced to a decimal, must either terminate or circulate.

Let $\frac{m}{n}$ be a fraction in its lowest terms.

Now, if n contains no factors except such as will divide some power of 10, the decimal will terminate. But the only numbers which will divide powers of 10 are powers of 2 or powers of 5. Now, if the denominator contains only powers of 2 and powers of 5, it is clear that some power of 10 will be arrived at by constantly affixing zero to the remainder in dividing, which will be divisible by the power of 2 or the power of 5 in the denominator.

Thus every fraction of the form $\frac{M}{2^p \times 5^q}$ will terminate whatever be the value of M, p, q, providing they are whole numbers.

Again, if n in $\frac{m}{n}$ contains other factors besides powers of 2 and 5, it cannot terminate, for those factors will not divide any power of 10.

Further, if $\frac{m}{n}$ does not terminate, it must circulate, for whatever be the value of n, there are only $n-1$ numbers less than it. And as the remainders, after dividing, must be less than n, there can only be $n-1$ remainders, without one of the remainders occurring again; $\therefore$ the frac-

tion $\frac{m}{n}$ must circulate in $n-1$ figures at most, if it does not terminate. This may be well illustrated by bringing $\frac{1}{7}$ to a decimal. The decimal is ·142857, and the remainders are 3, 2, 6, 4, 5. These remainders, with the figure 1 which we began with, form a complete set of all numbers below 7; and as all the numbers have been taken below 7, it is evident that the next number must be like one of the preceding, and in that case the fraction must circulate from the place where that remainder occurred before.

Therefore all fractions must either circulate or terminate when reduced to decimals.

In simplifying fractions there is a wide field for practice, and neatness in the methods of working can only be gained by experience. Let us work a few examples.

Ex. 1. Simplify $\dfrac{2\times\sqrt{1+\frac{1}{3}}\div\sqrt{1-\frac{1}{5}}}{5\times\sqrt{1+\frac{1}{3}}\times\sqrt{1-\frac{1}{5}}}$.

Simplifying the parts under roots first, we get

$$\frac{2\times\sqrt{\frac{4}{3}}\div\sqrt{\frac{4}{5}}}{5\times\sqrt{\frac{4}{5}}\times\sqrt{\frac{4}{5}}}$$

$$=\frac{2\times\dfrac{2}{\sqrt{3}}\times\dfrac{\sqrt{5}}{2}}{5\times\dfrac{2}{\sqrt{5}}\times\dfrac{2}{\sqrt{5}}}$$

$$=2\times\frac{2}{\sqrt{3}}\times\frac{\sqrt{5}}{2}\times\frac{1}{5}\times\frac{\sqrt{3}}{2}\times\frac{\sqrt{5}}{2}$$

$$=2\times\frac{1}{1}\times\frac{\sqrt{5}}{1}\times\frac{1}{5}\times\frac{1}{2}\times\frac{\sqrt{5}}{2}$$

$$=\frac{2\times\sqrt{5}\times\sqrt{5}}{5\times2\times2}=\frac{1}{2},\text{ Ans.}$$

Ex. 2. Reduce to a decimal correct to five figures the series

$$2\times\left\{\frac{1}{7}+\frac{1}{3}\times\frac{1}{7^3}+\frac{1}{5}\times\frac{1}{7^5}+\frac{1}{7}+\frac{1}{7^7}\times\ \ldots\right\}$$

$$\text{Here }\frac{1}{7}=\qquad 0\cdot1428571,$$

$$\frac{1}{7^2}=\frac{\cdot1428571}{7}=0\cdot0204081,$$

$$\frac{1}{7^3}=\frac{\cdot 0204081}{7}=0\cdot 0029154,$$
$$\frac{1}{7^4}=\frac{\cdot 0029154}{7}=0\cdot 0004164,$$
$$\frac{1}{7^5}=\frac{\cdot 0004164}{7}=0\cdot 0000594,$$
$$\frac{1}{7^6}=\frac{\cdot 0000594}{7}=0\cdot 0000084,$$
$$\frac{7}{7^7}=\frac{\cdot 0000084}{7}=0\cdot 0000012.$$

Since only five places are required, it is not necessary to add more terms of the series. Therefore we have

$$\frac{1}{7}=0\cdot 1428571,$$
$$\frac{1}{3}\times\frac{1}{7}=0\cdot 0009718,$$
$$\frac{1}{5}\times\frac{1}{7^5}=0\cdot 0000118,$$
$$\frac{1}{7}\times\frac{1}{7^7}=0\cdot 0000001,$$
$$\text{Sum}=0\cdot 1438408$$

Therefore the required decimal is . $\underline{2}$ $\cdot 2876816$

The rules for bringing circulating decimals to vulgar fractions may be thus proved.

(1) Let ·MMM, etc., be the decimal in which M recurs *ad infinitum.*

Let M contain p digits.

Let S be the value of the vulgar fraction equivalent to the decimal ·MMM, etc., *ad infin.*

$$\therefore\ S=\cdot MMM,\ \text{etc.}$$
$$\text{Then } 10^pS=M\cdot MMM,\ \text{etc.};$$
$$\therefore\ 10^pS-S=M\cdot MMM,\ \text{etc.}\ -\cdot MMM,\ \text{etc.}$$
$$=M;$$
$$\therefore\ S\,(10^p-1)=M;$$
$$\therefore\ S=\frac{M}{10^p-1}.$$

Now, whatever be the value of p, the denominator will contain the figure 9 only, and that figure will occur as many times as the number represented by p.

But p represents the number of digits which recur.

Hence the rule for reducing a pure circulating decimal to a vulgar fraction.

(2) Let ·PQQQ, etc., be a mixed circulating decimal where P contains m digits, Q contains n digits, recurring *ad infin.*

Let S be the vulgar fractions to which it is equivalent.

$$\text{Then } S = \cdot PQQQ, \text{ etc.}$$
$$\text{And } 10^{m+n}S = PQ\cdot QQQ, \text{ etc.}$$
$$\text{And } 10^{m}S = P\cdot QQQ, \text{ etc.}$$
$$\therefore (10^{m+n} - 10^{m})S = PQ\cdot QQQ, \text{ etc.} - P\cdot QQQ, \text{ etc.}$$
$$= PQ - P;$$
$$\therefore S = \frac{PQ - P}{10^{m}(10^{n} - 1)}.$$

Hence the rule for reducing a *mixed* circulating decimal to its equivalent vulgar fraction.

(*c*) Rule of Three and its Applications.

In working out examples in proportion or rule of three, care should be taken in cancelling. The quantities in the second and third term should be written as the numerator, and those in the first term as the denominator of a fraction, without any alteration at first; and as the common factors are cancelled out, the numbers should not be crossed out with a line, but re-written in the form of a fraction, with unity in the place of those numbers which have been cancelled out. Thus, suppose we have the proportion

$$\left.\begin{array}{r} 27 : 121 \\ 16 : 9 \\ 11 : 3 \end{array}\right\} :: 64,$$

then the result is

$$\frac{64 \times 3 \times 9 \times 121}{11 \times 16 \times 27} = \frac{4 \times 1 \times 1 \times 11}{1 \times 1 \times 1} = 44, \text{ Ans.}$$

The rule known as 'first principles' is often useful in preparing examples for the application of 'rule of three.' For example, one pipe can fill a cistern in two hours, and another in four hours, while a third can empty it in three hours. In what time will the cistern be full if all are open together.

First puts in $\frac{1}{2}$ of cisternful in one hour.
Second ,, $\frac{1}{4}$,, ,, ,, ,, hour.
Third lets out $\frac{1}{3}$,, ,, ,, ,, hour.

$\therefore$ If all are open together, quantity in one hour put in $= \frac{1}{2} + \frac{1}{4} - \frac{1}{3} = \frac{6}{12} + \frac{3}{12} - \frac{4}{12}$
$= \frac{5}{12}$ in 1 hour.

Now, if this cistern is $\frac{5}{12}$ full in 1 hour, what time will it take to fill it?

$$\therefore \tfrac{5}{12} : 1 :: 1 : \tfrac{12}{5} = 2 \text{ hrs. } 24 \text{ mins.}$$

The following is an example of the same rule, used *instead* of rule of three:—

If 10 men build a wall 10 feet high, 2 feet thick, and 140 feet long in 12 days, how many days are necessary for 25 men to build a wall 1 foot thick, 12 feet high, 210 feet long?

In 12 days 10 men build a wall 10 feet $\times$ 2 feet $\times$ 140 feet $= 20 \times 140 = 2800$ c. f.

$\therefore$ In 1 day 10 men build a wall of $\dfrac{2800}{12}$ c. f.

$\therefore$ In 1 day 1 man builds a wall of $\dfrac{2800}{12 \times 10}$ c. f.

$\therefore$ In $\dfrac{12 \times 10}{2800}$ days 1 man builds a wall of 1 c. f.

Now, the second wall contains $1 \times 12 \times 210$ feet $= 2520$ c. f.

$\therefore$ In $\dfrac{12 \times 10}{2800} \times 2520$ days 1 man builds 2520 c. f.

And in $\dfrac{12 \times 10 \times 2520}{2800 \times 25}$ days 25 men build 2520 c. f., or the second wall.

But,

$$\frac{12 \times 10 \times 2520}{2800 \times 25} \text{ days} = \frac{3 \times 1 \times 252}{7 \times 25} = \frac{3 \times 1 \times 36}{25} \text{ days}$$
$$= \frac{108}{25} \text{ days} = 4\tfrac{8}{25} \text{ days.}$$

(*d*) Square Root.

The ordinary rule for square root must be mastered, and it will be well for the student to accustom himself to the method of finding the square root of a number by means of factors, as shown in the previous chapter. The method of factors is frequently useful, and especially so in cases like the following:—

Find the square root of $\frac{675}{768}$.

We may, if we choose, reduce this fraction to a decimal, and then extract the square root in one process; but it is usually better, in extracting the square root of a fraction, to extract the square root of the numerator and denominator separately, and if the answer is required to be in a decimal, to reduce the result to a decimal afterwards. In case the two parts of the fraction are not *each* complete squares, it is always advisable to find the factors, as examples are sometimes given in which both parts of the fraction are complete squares when a common factor is taken out. Thus,

$$\frac{675}{768}=\frac{225\times 3}{256\times 3}=\frac{225}{256},$$

the square root of which is $\frac{15}{16}$.

Examples of this kind are by no means uncommon.

In extracting the square root of a number which is not a complete square, the decimal can neither terminate nor circulate.

It cannot terminate, for if it did, the number would be a square number.

It cannot circulate, for the remainders increase without limit, and therefore a remainder having the same value as a previous remainder is impossible.

It is often possible to form an idea by inspection whether a number is a square number or not. The following hints may be interesting and useful:—

(1) If the units digit of a number is 2, 3, 7, or 8, it cannot be a square number.

(2) If the last digit is zero, it cannot be a square number unless the last figure but one is also zero; or if a number has an odd number of zeros on the right hand, it cannot be a square number.

(3) If a number has an odd number of decimal places, it cannot be a square number.

(4) If a number terminate with 5, it is not a square number unless the next figure is 2.

(5) If a number terminates with two figures the same, it cannot be a square number unless they are two 4's or two ciphers.

(6) If the last figure of a number is odd, it cannot

be a square number unless the preceding figure is even.

(7) If the last figure of a number is even, it cannot be a square number unless the last figure but one is odd, except the last figure is 4.

In extracting the square root, we may, after finding one figure more than half the number of figures in the root, obtain the remainder by division only.

Thus, suppose we require the square root of 2 to five places of decimals :—

	2·0000(1·414
	1
24	100
	96
281	400
	281
2824	11900
	11296
2828	60400000(21
	5656
	3840
	2828
	1012

∴ The square root of 2 is 1·41421.

A proof of this rule may be given.

Let M be a number whose square root, consisting of $2n+1$ digits, is required.

Let a be the first $n+1$ digits of the root with n ciphers affixed.

Let x be the remaining part of the root which will take the place of the n ciphers when the root is complete.

$$\text{Then } \sqrt{M}=a+x;$$
$$\therefore\ M=a^2+2ax+x^2;$$
$$\therefore\ M-a^2=2ax+x^2;$$
$$\therefore\ \frac{M-a^2}{2a}=x+\frac{x^2}{2a}.$$

That is, $M-a^2$ (which is the remainder after the first $n+1$ digits are found), divided by $2a$, will give the rest of the root required, (x) increased by $\frac{x^2}{2a}$.

But since x contains n digits, x^2 cannot have more than $2n$ digits at most.

But by hypothesis a has $2n+1$ digits altogether, and $2a$ has $2n+1$ digits at least;

$$\therefore x^2 \text{ is less than } 2a;$$

$$\therefore \frac{x^2}{2a} \text{ is a proper fraction,}$$

$$\text{or } \frac{x^2}{2a} \angle 1.$$

That is, if $\frac{M-a^2}{2a}$ be taken for x instead of $x+\frac{x^2}{2a}$, the error is less than unity.

$\therefore$ If $n+1$ digits be obtained by the usual process, n digits may be obtained by division only.

We have deviated a little from the order in which the subjects are given in the syllabus. We passed over Interest, Present Value, Discount, and Annuities, to the rules for the extraction of square root, our object being to leave these subjects until after we have taken up 'The Nature and Use of Logarithms,' and we advise the student to take up the subjects in the same order, as a knowledge of logarithms will be found valuable in working out examples in the above subjects.

(*e*) THE NATURE AND USE OF LOGARITHMS.

Although this subject is included under the head of arithmetic, the questions which have been given the last two or three years show that a much more extensive knowledge of logarithms is required than that which is usually given in arithmetical treatises, and we advise the student to master the subject as it is given in Todhunter's *Algebra for Colleges and Schools*, and in *Plane Trigonometry*, by the same author, chap. xi. sections 151 to 163.

There is little difficulty in the use of logarithms when their *nature* is thoroughly understood.

A logarithm is thus defined:—'The logarithm of a number to a given base is the index of the power to which the base must be raised to be equal to the number.'

Let us illustrate this by taking 2 as a base.

$2^1=2$	$2^7=128$
$2^2=4$	$2^8=256$
$2^3=8$	$2^9=512$
$2^4=16$	$2^{10}=1024$
$2^5=32$	$2^{11}=2048$
$2^6=64$	$2^{12}=4096$, etc.

Thus, with 2 as base, 1 is the log of 2, 2 is the log of 4, 5 is the log of 32, and so on.

Any number may be used as the base of logarithms, and these logarithms will answer for calculation. There are, however, advantages in taking 10 as the base, and it is to this base that ordinary logarithms are calculated. We shall refer to the advantages of this base hereafter, and in the meantime we work a few examples of the use of logarithms, having 2 as the base.

Suppose we want to multiply 32 by 64.

The logarithm of their product will be the *sum* of the logarithms of the numbers themselves.

The log of 32 is 5, the log of 64 is 6; $\therefore$ the log of 32×64 will be $6+5$, *i.e.* 11.

And by reference to the table above, 11 is the log of 2048;

$$\therefore\ 32\times 64=2048.$$

Again, to divide 4096 by 512.

The logarithm of their quotient will be the *difference* of the logarithms of the numbers themselves.

The log of 4096 is 12, the log of 512 is 9; $\therefore$ log of quotient $=12-9=3$.

And 3 is the log of 8; $\therefore\ 4096\div 512=8$.

Again, suppose we want the cube root of 4096.

The log of the cube root will be one-third of the log of the number.

$$\text{But log of } 4096=12,$$
$$\text{and one-third of } 12=4;$$
$$\therefore\ 4 \text{ is the log of the cube root};$$
$$\therefore\ \sqrt[3]{4096}=16.$$

If the sixth root had been required, we should have

divided the log of the given number by 6; if the square root, by 2, and so on.

Again, suppose we want the fourth power of 8.

The log of the fourth power of 8 will be four times the log of 8 itself.

But the log of $8=3$; $\therefore$ the log of the fourth power of 8 will be $3\times 4=12$.

But 12 is the log of 4096; $\therefore$ $8^4=4096$.

It will be necessary for the student to provide himself with a book of tables of logs. A cheap and good one is published by W. Stewart & Co., London.

We will assume that the student has provided himself with a set of tables such as we have mentioned. All the above examples may be worked out with logs to the base 10, as given in them.

For example, the product of 32 and 64 may be thus found:—

$$\begin{aligned} \text{Log of } 32 \text{ to base } 10 &= 1{\cdot}50515 \\ \text{log of } 64 \text{ to base } 10 &= 1{\cdot}80618 \\ \hline \therefore \text{log of } 64\times 32 &= 3{\cdot}31133 \end{aligned}$$

Taking the decimal part of the log, we find by the tables that the natural number corresponding to it is 20480.

And the integral part being 3, the number corresponding with the log $3{\cdot}31133$ will be $2048{\cdot}0$, as before.

If we have the logarithm of one number to any base, a large number of other logarithms may be found from it. For example, if we have the logarithm of 2 to the base 10, we may find the log of any power of 2 from it.

$$\text{Thus log } 2 \text{ to base } 10 = {\cdot}30103;$$
$$\therefore \log 2^2 \text{ or } \log 4 = {\cdot}30103\times 2;$$
$$\therefore \log 2^3 \text{ or } \log 8 = {\cdot}30103\times 3, \text{ and so on.}$$

And if we have log 3, we may find the log of any number which contains no factors except 2 and 3. For example,

$$\text{Log } 6 = \log 2 + \log 3,$$
$$\log 54 = 3\log 3 + \log 2, \text{ and so on.}$$

The logs of prime numbers are usually found by means of a series. The proof of this series is beyond the limits of our subject at present, but we give it, and show a few examples of its use.

Let N be any number.

$$\text{Then } \log(N+1) = \log N + 2\mu\left\{\frac{1}{2N+1} + \frac{1}{3}\cdot\frac{1}{(2N+1)^3} + \frac{1}{5}\cdot\frac{1}{(2N+1)^5} + \frac{1}{7}\cdot\frac{1}{(2N+1)^7} + \text{etc.}\right\}$$

μ is a multiplier, of which the value $= \cdot 43429448$.

To find log 2 to the base 10.

Let $N = 1$. Then $N + 1 = 2$;

$$\therefore \log 2 = \log 1 + 2\mu\left\{\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3^3} + \frac{1}{5}\cdot\frac{1}{3^5} + \text{etc.}\right\}$$

And since $\log 1 = 0$;

$$\therefore \log 2 = 2\mu\left\{\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3^3} + \frac{1}{5}\cdot\frac{1}{3^5} + \frac{1}{7}\cdot\frac{1}{3^7} + \text{etc.}\right\}$$

$$= 2 \times \cdot 43429448\left\{\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3^3} + \frac{1}{5}\cdot\frac{1}{3^5} + \text{etc.}\right\}$$

$$= \cdot 30103.$$

The series being infinite, its *exact* value cannot be found; but by increasing the number of terms, it may be made as nearly accurate as we please.

Again, let $N = 2$.

$$\text{Then } \log 3 = \log 2 + 2\mu\left\{\frac{1}{5} + \frac{1}{3}\cdot\frac{1}{5^3} + \frac{1}{5}\cdot\frac{1}{5^5} + \text{etc.}\right\}$$

$$= \cdot 301030 + \cdot 176091$$

$$= \cdot 477121.$$

Having now obtained logs of 2 and 3, we may find many others from them.

For example, $\log 4 = 2 \times \log 2$
$= 2 \times \cdot 301030$
$= \cdot 602060$,

$\log 5 = \log(\frac{10}{2}) = \log 10 - \log 2$
$= 1 - \cdot 301030$
$= \cdot 698970$,

$\log 6 = \log 2 + \log 3 = \cdot 301030 + \cdot 477121$
$= \cdot 778151$,

$\log 8 = \log(2^3) = 3 \times \log 2 = 3 \times \cdot 301030$
$= \cdot 903090$,

$\log 9 = \log(3^2) = 2 \times \log 3 = 2 \times \cdot 477121$
$= \cdot 954242$,

$\log 10 = 1$, for it is the base itself,

$\log 12 = \log 3 + \log 4 = \log 3 + 2 \log 2$
$= \cdot 477121 + \cdot 602060$
$= 1 \cdot 079181$.

We have thus found all the logs from 1 to 12, except

7 and 11. These must be found by the series. Thus, let N = 6.

$$\text{Log } 7 = \log 6 + 2\mu\left\{\frac{1}{13} + \frac{1}{3}\cdot\frac{1}{13^3} + \frac{1}{5}\cdot\frac{1}{13^5} + \text{etc.}\right\}$$
$$= \cdot778151 + \cdot066947$$
$$= \cdot845098.$$

Again, let N = 10. We get

$$\log 11 = \log 10 + 2\mu\left\{\frac{1}{21} + \frac{1}{3}\cdot\frac{1}{21^3} + \frac{1}{5}\cdot\frac{1}{21^5} + \text{etc.}\right\}$$
$$= 1 + \cdot041392$$
$$= 1\cdot041392.$$

We may, with the logs thus found, calculate an unlimited number of others.

For example, $\log 14 = \log 2 + \log 7,$
$\log 15 = \log 3 + \log 5,$
$\log 16 = \log (2^4) = 4 \log 2,$
$\log 18 = \log 2 + \log 9$
$= \log 2 + 2 \log 3,$
$\log 20 = \log 2 + \log 10 = \log 2 + 1,$
$\log 21 = \log 3 + \log 7,$
$\log 22 = \log 2 + \log 11,$
$\log 24 = \log 3 + 3 \log 2,$
$\log 25 = 2 \log 5,$
$\log 27 = 3 \log 3,$
$\log 28 = \log 4 + \log 7,$
$\log 30 = \log 3 + \log 10 = 1 + \log 3.$

Logarithms of 13, 17, 19, 23, 29 must be found from the series; log 26 will be log 2 + log 13.

We now proceed to work out a few examples on the method of finding the logs of numbers which have been given in the examinations.

(1) Given $\log 648 = 2\cdot81157501,$
$\log 864 = 2\cdot93651374.$

To find log 108 (1872).

$$864 = 32 \times 27 = 2^5 \times 3^3,$$
$$648 = 8 \times 81 = 2^3 \times 3^4.$$

Let $x = \log 2$, $y = \log 3$.

Then $5 \log 2 + 3 \log 3 = \log 864 = 2\cdot9365137.$

And $3 \log 2 + 4 \log 3 = \log 648 = 2\cdot8115750,$

or $5x + 3y = 2\cdot9365137,$
$3x + 4y = 2\cdot8115750.$

By solving these equations we get

$$\log 2 = \cdot 3010300,$$
$$\log 3 = \cdot 4771213.$$

Now $108 = 27 \times 4 = 3^3 \times 2^2$;

$$\therefore \log 108 = 3 \log 3 + 2 \log 2$$
$$= 3 \times \cdot 4771213 + 2 \times \cdot 3010300$$
$$= 1 \cdot 4313639 + \cdot 6020600$$
$$= 2 \cdot 0334239, \text{ Ans.}$$

(2) Given $\log 2 = \cdot 301030$ and $\log 3 = \cdot 477121$, find the logs of $\cdot 00625$, of $\frac{1}{24}$, and of $(\cdot 0003)$.

Also find approximately the value of x which satisfies $2^x = 5$ (1873).

First find log 5.

$$10 = 5 \times 2; \therefore \log 10 = \log 5 + \log 2;$$
$$\therefore \log 5 = \log 10 - \log 2.$$

But $\log 10 = 1$, $\log 2 = \cdot 301030$;

$$\therefore \log 5 = 1 - \cdot 301030 = \cdot 698970.$$

Now, if log 625 were required, we should write

$$625 = 5^4;$$
$$\therefore \log 625 = 4 \log 5 = 4 \times \cdot 698970$$
$$= 2 \cdot 795880;$$
$$\therefore \log 62 \cdot 5 = 1 \cdot 795880,$$
$$\log 6 \cdot 25 = \cdot 795880,$$
$$\log \cdot 625 = \bar{1} \cdot 795880,$$
$$\log \cdot 0625 = \bar{2} \cdot 795880,$$
$$\log \cdot 00625 = \bar{3} \cdot 795880.$$

$$\frac{1}{24} = \frac{1}{3 \times 8} = \frac{1}{3 \times 2^3} = \log 1 - \log 3 - 3 \log 2$$
$$- \cdot 477121 - \cdot 903090$$
$$= -1 \cdot 380211.$$

But, as the mantissa or decimal part must be always positive, we write

$$-1 \cdot 380211 \text{ as } 2 - 1 \cdot 380211 - 2$$
$$= \bar{2} \cdot 619789, \text{ Ans.}$$

Again $(\cdot 0003)^5$, if $\log 3 = \cdot 477121$,

$$\log \cdot 0003 = \bar{4} \cdot 477121;$$
$$\therefore \log (\cdot 0003)^5 = \bar{4} \cdot 477121 \times 5$$
$$= \overline{18} \cdot 385605.$$

Again, if $2^x = 5$, $x \log 2 = \log 5$;

$$\therefore x = \frac{\log 5}{\log 2}.$$

But log $5 = \cdot 698970$ (see above);

$$\therefore x = \frac{\cdot 698970}{\cdot 301030} = 2 \cdot 32.$$

(3) Having given

$$\log 193 \cdot 06 = 2 \cdot 2856923,$$
$$\log 19307 = 4 \cdot 2857148,$$

find the 7th root of 100 (1877).

$$\text{Log of } 100 = 2;$$

$$\therefore \log \sqrt[7]{100} = \tfrac{2}{7} = \cdot \dot{2}85714\dot{2}85$$
$$\log 1 \cdot 9306 = \underline{\cdot 285692300}$$
$$\text{Difference,} \qquad 21985$$

for 7 decimal places of the logs.

$$\text{But } \log 1 \cdot 9307 = \cdot 2857148$$
$$\log 1 \cdot 9306 = \underline{\cdot 2856923}$$
$$\text{Difference,} \qquad 225$$

$$\therefore \text{ number required} = 1 \cdot 9306 \tfrac{219 \cdot 85}{225}.$$

$$\begin{array}{r|l} 225)\,219 \cdot 85 & (977 \\ \underline{2025} & \\ \cdot 1735 & \\ \underline{1575} & \\ \cdot 160 & \end{array}$$

$$\therefore \text{ the 7th root of } 100 = 1 \cdot 9306977.$$

The following are the more important theorems in logarithms, and the proofs must be carefully studied and mastered:—

(1) The log of 1 is 0, whatever the base may be.

(2) The log of the base is always unity, whatever the base may be.

(3) Log $mn = \log m + \log n$.

(4) Log $\frac{m}{n} = \log m - \log n$.

(5) Log $m^r = r \log m$.

(6) In the common system of logarithms, the characteristic may be told by inspection.

(7) The change of a logarithm is approximately proportional to the change of a number, *i.e.*

$$\log (n+a) - \log n = \frac{\mu d}{n}.$$

(See Todhunter's *Trigonometry*, sec. 177, for this proof.)

(8) The log of a to the base b, multiplied by the log of b to the base $a = 1$.

The log sines, log cosines, log tangents, etc., usually

found in tables of logarithms, are the logarithms of the natural sines, natural cosines, and tangents. As some of the trigonometrical functions of an angle never exceed unity, their logarithms would have -1 as the characteristic. On this account, their logarithms are increased by 10, so that the characteristic may be positive.

The following example will show how the logarithms of trigonometrical functions are calculated.

Take angle 30°.

$$\text{Nat. } \sin 30° = \tfrac{1}{2} = \cdot 5.$$

$$\text{The log of } \cdot 5 = \bar{1}\cdot 6989700;$$

$$\therefore \log \sin 30° = \bar{1}\cdot 6989700 + 10$$
$$= 9\cdot 6989700.$$

$$\cos 30° = \frac{\sqrt{3}}{2};$$

$$\therefore \log \cos 30° = \tfrac{1}{2} \log 3 - \log 2$$
$$= \frac{\cdot 477121}{2} - \cdot 3010300$$
$$= -\cdot 0624694 + 10 = 9\cdot 9375306,$$

$$\tan 30° = \frac{\sin 30}{\cos 30};$$

$$\therefore \log \tan 30° = \log \sin 30 - \log \cos 30$$
$$= 9\cdot 6989700 - 9\cdot 9375306$$
$$= -\cdot 2385606 + 10;$$

$$\therefore \log \tan 30 = 9\cdot 7614394.$$

In the same manner the logarithms of the other trigonometrical functions are calculated.

(*d*) Simple and Compound Interest.

Passing over the arithmetical rules for Simple Interest, knowing that all who have passed the Matriculation will be acquainted with them, we proceed to consider the algebraic method of working examples in this subject.

The following symbols are usually used in interest:—

P = Principal in pounds.

n = Number of years for which interest is calculated.

r = Interest on £1 for one year.

M = Amount.

R = Amount of £1 for one year;

$\therefore R = 1 + r$.

The chief formula in simple interest is $M = P + Pnr$.

A proof of this formula must be thoroughly mastered from some standard work on the subject.

Any three of the four quantities P, n, r, M being given, the other may be calculated.

$$\text{Thus } P=\frac{M}{1+nr},\ n=\frac{M-P}{Pr},\ r=\frac{M-P}{Pn}.$$

Care must be taken to remember that r is not the rate per cent., but the interest on £1 for one year. Thus, if the interest is

4 per cent., $r=£\cdot04$, and $R=£1\cdot04$.
If 5 " " $r=£\cdot05$, and $R=£1\cdot05$.
If $2\frac{1}{4}$ " " $r=£\cdot0225$, and $R=£1\cdot0225$,

and so on.

The corresponding formula in compound interest is $M=PR^n$, from which

$$P=\frac{M}{R^n},\ n=\frac{\log M-\log P}{\log R},\ R=\left(\frac{M}{P}\right)^{\frac{1}{n}},$$

$$\text{or } \log R=\frac{1}{n}(\log M-\log P).$$

If, also, I be the interest on any amount,

$$I=M-P=PR^n-P=P(R^n-1).$$

The above formulæ apply only when the interest is paid *yearly*. If paid more frequently, it must be *modified* as follows. If paid twice a year, $\frac{r}{2}$ will be the interest, up to the time of the first payment, on £1. And in n years there will be $2n$ payments; $\therefore$ r will thus become $\frac{r}{2}$ and n will become $2n$; $\therefore$ £P in n years will amount to £$P\left(1+\frac{r}{2}\right)^{2n}$, or $M=P\left(1+\frac{r}{2}\right)^{2n}$.

Similarly, if payable *quarterly*,

$$M=P\left(1+\frac{r}{4}\right)^{4n}.$$

If paid *q times a year*,

$$M=P\left(1+\frac{r}{q}\right)^{qn}.$$

We may calculate the advantage of half-yearly and quarterly payments of compound interest.

Amount of £1 by yearly interest $=1+r$,

amount of £1 by half-yearly interest $=\left(1+\frac{r}{2}\right)^2$;

$$\therefore \text{advantage} = \left(1+\frac{r}{2}\right)^2 - (1+r)$$
$$= \left(1+r+\frac{r^2}{4}\right) - (1+r)$$
$$= \frac{r^2}{4}.$$

Thus, at 5 per cent., the advantage on £1 would be

$$\frac{\cdot05^2}{4} = \frac{\cdot0025}{4} = £\cdot000625;$$

∴ on £10,000 it would amount to £6, 5s. 0d. only. The advantage, however, increases greatly in the course of years.

Again, by quarterly payments of interest the amount would be $\left(1+\frac{r}{4}\right)^4$, and the advantage

$$\left(1+\frac{r}{4}\right)^4 - (1+r)$$
$$= 1 + r + \frac{3r^2}{8} + \frac{r^3}{16} + \frac{r^4}{256} - (1+r)$$
$$= \frac{3r^2}{8} + \frac{r^3}{16} + \frac{r^4}{256}.$$

But since r is a small fraction, we may take this as nearly equal to $\frac{3r^2}{8}$.

At 5 per cent. this would be

$$\frac{3 \times \cdot0025}{8} = £\frac{\cdot0075}{8} = £\cdot0009375,$$

or on £10,000 the advantage would be but £9, 7s. 6d. In practice, compound interest is only reckoned for an integral number of years; for a fractional part of a year, simple interest is allowed.

It is interesting to calculate the number of years at which money will double itself at compound interest at different rates per cent.

Suppose P be the amount given.

Then P is to double itself in n years—that is, it is to become 2P;

$$\therefore M = 2P; \therefore 2P = PR^n;$$
$$\therefore 2 = R^n; \therefore \log 2 = n \log R;$$
$$\therefore n = \frac{\log 2}{\log R} = \frac{\log 2}{\log(1+r)}.$$

Now, if the rate be 5 per cent.,

$$n = \frac{\log 2}{\log 1{\cdot}05} = 14{\cdot}206 \text{ years.}$$

If the rate be 10 per cent.,

$$n = \frac{\log 2}{\log 1{\cdot}1} = 7{\cdot}272 \text{ years.}$$

If the rate be $2\frac{1}{2}$ per cent.,

$$n = \frac{\log 2}{\log 1{\cdot}025} = 28{\cdot}071 \text{ years.}$$

By a similar process we may find in how many years £100 will amount to £1000 at 5 per cent.

$$1000 = 100\,(\mathrm{R})^n,$$

$$10 = \mathrm{R}^n = 1\,;$$

$$\therefore\ n = \frac{\log 10}{\log 1{\cdot}05} = \frac{1}{{\cdot}02118}$$

$$= 47{\cdot}2 \text{ years.}$$

After studying progressions, the student will observe that at simple interest the amount of P for n years will be the nth term of an arithmetic series whose first term is P, and common difference Pr; and that at compound interest the amount of P for n years will be the nth term of a geometric series whose first term is P and common ratio $1+r$ or R.

(e) Present Value, Discount, and Annuities.

The difference between true discount and interest must be mastered.

Taking D as discount, P as present value, M the amount, and r and R as before.

At simple interest $\mathrm{M} = \mathrm{P}\,(1+nr)$;

$$\therefore\ \mathrm{P} = \frac{\mathrm{M}}{1+nr}, \text{ and } \mathrm{D} = \mathrm{M} - \mathrm{P} = \mathrm{M} - \frac{\mathrm{M}}{1+nr}$$

$$= \frac{\mathrm{M}\,(1+nr) - \mathrm{M}}{1+nr} = \frac{\mathrm{M}nr}{1+nr}.$$

The discount is always less than the interest on the amount for the same time.

Let I be the interest on a sum M, on which D is the discount.

Then $\mathrm{M} + \mathrm{I} = \mathrm{M}\,(1+nr)$;

$$\therefore\ \frac{\mathrm{M}+\mathrm{I}}{\mathrm{M}} = 1 + nr;\ \therefore\ \frac{1}{1+nr} = \frac{\mathrm{M}}{\mathrm{M}+\mathrm{I}};$$

$$\therefore \frac{M}{1+nr} = M \cdot \frac{M}{M+I}. \quad \text{But } P = \frac{M}{1+nr};$$

$$\therefore P = M \cdot \frac{M}{M+I}.$$

$$\text{But } D = M - P = M - M \cdot \frac{M}{M+I}$$

$$= \frac{MI}{M+I};$$

$$\therefore \frac{1}{D} = \frac{M+I}{MI} = \frac{1}{I} + \frac{1}{M}.$$

And as $\frac{1}{D}$ is equal to two fractions whose denominators are I and M. D is evidently less than either of them; $\therefore$ D is less than I.

If true discount be reckoned with compound interest,

$$M = P(1+r)^n; \therefore P = \frac{M}{(1+r)^n}.$$

In this case also the discount may be shown to be less than the interest by the same method as the above.

For studying annuities, we recommend Wood's *Algebra* by Lund, or Todhunter's *Algebra for Colleges and Schools*.

The proofs of the following formulæ must be well understood.

If A be the amount of an annuity, n the number of years, r the interest on £1 for one year, R the amount of £1 for one year.

Then the amount M of an annuity left unpaid for n years will be—

(1) At simple interest,

$$M = nA + \frac{n(n-1)}{2} rA.$$

(2) At compound interest,

$$M = \frac{R^n - 1}{R - 1} A.$$

The present value P of an annuity A to continue n years is—

(1) At simple interest,

$$P = \frac{nA + \frac{1}{2}n(n-1)rA}{1+nr}.$$

(2) At compound interest,

$$P = \frac{R^n - 1}{R^n (R-1)} A.$$

If the annuity continues for ever, n is infinite.

Then at simple interest we may write the formula,

$$P = \frac{A + \frac{1}{2}(n-1)rA}{\frac{1}{n} + r},$$

which is infinite; and at compound interest,

$$P = \frac{A\left(1 - \frac{1}{R^n}\right)}{R-1}.$$

But if n is infinite,

$$\frac{1}{R^n} = 0; \therefore P = \frac{A}{R-1} = \frac{A}{r}.$$

If the annuity commences at the end of p years, and then continues q years, its present value,

$$P = \frac{A}{R-1} \cdot \left(\frac{1}{R^p} - \frac{1}{R^{(p+q)}}\right).$$

If it commences at the end of p years, and continues for ever,

$$P = \frac{A}{R-1} \cdot \frac{1}{R^p} = \frac{A}{r \cdot R^p}.$$

The reader will note that the amount of an annuity for n years at simple interest is the sum of n terms of an arithmetical progression whose first term is A, and common difference rA.

$$\text{Thus, } M = nA + \frac{n(n-1)rA}{2}$$

$$= \left\{ 2A + (n-1)rA \right\} \frac{n}{2};$$

and at compound interest it is the sum of a geometric progression whose first term is A, and common ratio R or $1+r$. Also, that if the annuity continue for ever, its amount will be the sum of the same series *ad infinitum*.

We conclude by working a few questions in these subjects which have been given at the examinations.

1. An annuity is to begin n years hence, and continue for ever. Find its present value, and show that its present value : its value in n years hence :: $(1+r)^{-n} : 1$ (1871).

Its value, if it began at present and continued for ever, would be $\frac{A}{r}$.

Its value for n years will be $A \cdot \frac{R^n - 1}{R^n(R-1)}$;

$\therefore$ its present value will be $\frac{A}{r} - A \cdot \frac{R^n - 1}{R^n(R-1)}$

$$= \frac{A}{r} - A \cdot \frac{(1+r)^n - 1}{(1+r)^n r} = \frac{A(1+r)^n - A(1+r)^n + A}{(1+r)^n r}$$

$$= \frac{A}{(1+r)^n r}.$$

Its value in n years will be $\frac{A}{r}$;

$$\therefore \text{pres. val. : val. in } n \text{ years} :: \frac{A}{(1+r)^n r} : \frac{A}{r}$$

$$:: \frac{1}{(1+r)^n} : 1$$

$$:: (1+r)^{-n} : 1.$$

2. The mantissæ of the logs of 2, 3, 11 being ·301030, ·477121, ·041393, find in how many years the population of a country will be doubled when the birth-rate at the end of each year is $\frac{1}{30}$th, and the death-rate $\frac{1}{40}$th of the population at the beginning of that year (1874).

Let p be the population at the beginning of a year. At the end of first year it will be

$$p + \frac{p}{30} - \frac{p}{40} = p\,\frac{121}{120}.$$

At the end of second year it will be $p\left(\frac{121}{120}\right)^2$.

At the end of n years it will be $p\left(\frac{121}{120}\right)^n$.

Suppose the population to be doubled in n years;

$$\therefore p\left(\frac{121}{120}\right)^n = 2p, \text{ or } \left(\frac{121}{120}\right)^n = 2;$$

$$\therefore n \log 121 - n \log 120 = \log 2;$$

$$\therefore n\{2 \log 11 - (3 \log 2 + \log 3 + \log 5)\} = \log 2;$$

$$\therefore n\{2{\cdot}082786 - ({\cdot}903090 + {\cdot}477121 + {\cdot}698970)\}$$

$$= {\cdot}301030;$$

$$\therefore n = \frac{{\cdot}301030}{{\cdot}003605} = 83 \text{ years.}$$

3. A person starts with a certain capital, which produces him 4 per cent. per annum, compound interest. He spends every year a sum equal to twice the original interest on his capital. Find in how many years he will be

ruined, having given log $2 = \cdot301030$, log $13 = 1\cdot1139434$ (1867).

Let P be his capital. Then his interest will be $\frac{P}{25}$, and his expenditure $\frac{2P}{25}$.

Now, when the amount of P at compound interest becomes equal to the value of an annuity of $\frac{2P}{25}$ per annum, he will be ruined.

Amount of P at compound interest in n years $= P . R^n = P . (1\cdot04)^n$; and value of the annuity

$$\frac{\frac{2P}{25}(R^n - 1)}{R - 1} = \frac{\frac{2P}{25}(1\cdot04^n - 1)}{\cdot04}.$$

We have therefore to find n from the equation

$$P\,(1\cdot04)^n = \frac{2P}{25}\,\frac{(1\cdot04^n - 1)}{\cdot04};$$

$$\therefore\ 1\cdot04^n \times \cdot04 = \frac{2}{25}\,(1\cdot04^n - 1);$$

$$\therefore\ 1\cdot04^n \times \cdot04 = \cdot08\,(1\cdot04^n - 1);$$

$$\therefore\ 1\cdot04^n \times \cdot01 = 2\,(1\cdot04^n - 1);$$

$$\therefore\ 1\cdot04^n = 2;$$

$$\therefore\ n \log 1\cdot04 = \log 2.$$

But $104 = 8 \times 13 = 2^3 \times 13$;

$$\therefore \log 104 = 3 \log 2 + \log 13$$
$$= \cdot9030900 + 1\cdot1139434 = 2\cdot0170334;$$
$$\therefore\ \log 1\cdot04 = \cdot0170334;$$
$$\therefore\ n \times \cdot0170334 = \cdot3010300;$$
$$\therefore\ n = \frac{\cdot301030}{\cdot0170334} = 17\cdot6 \text{ years.}$$

4. A man borrows £1000, which is to be repaid with compound interest at five per cent. He is to pay as much of the capital each year as he pays interest for that year, and at the end of ten years he is to pay the balance. What will he have to pay as his balance at the end of the ten years?

Suppose the amount borrowed to be £P. Amount at end of one year $= P + Pr = P(1 + r)$. But he pays $2Pr$; $\therefore$ at the beginning of the second year he owes $P - Pr = P(1 - r)$.

At the end of the second year he owes, after making

his payment, P $(1-r)(1-r) = P(1-r)^2$. And at the end of the ten years, after his annual payment, he owes P $(1-r)^{10}$.

Substituting, we get his balance,

$$1000\,(\cdot 95)^{10} = 1000.$$

$$\text{But } \log \cdot 95 = \bar{1}\cdot 9777236\,;$$

$$\therefore\ 10 \log \cdot 95 = \bar{1}\cdot 777236.$$

But $\bar{1}\cdot 777236$ is the log of $\cdot 59873$;

$\therefore$ his balance = £598·73.

II. ALGEBRA.

THIS section is divided into seven parts:—

(*a*) The ordinary rules of algebra.
(*b*) Reduction and manipulation of algebraic fractions.
(*c*) Ratio, proportion, and variation.
(*d*) Permutations and combinations.
(*e*) Arithmetical and geometrical progressions.
(*f*) Simple and quadratic equations.
(*g*) Determination of common factors.

The first part of this section needs but little comment. We may, however, draw the attention of the student to two points—1st. Factors must be mastered; and, 2d. The rules for changing signs in a fraction must be well understood. A good knowledge of these points often decreases labour at the time of the examination, and many examples may be found, in looking over the questions of the last few years, which show the very great importance of the hints here given. The student who is well acquainted with factors will not fail to earn credit in these examinations.

Examples like the following are not uncommon:—

$$1.\quad \left.\begin{aligned} \frac{x-y+1}{x} &= \frac{1}{a-b} \\ \frac{y-x-1}{y} &= \frac{1}{a+b} \end{aligned}\right\} \begin{aligned} &(a) \\ &(b). \end{aligned}$$

Multiply the numerator of each side of (*b*) by -1, and it becomes $\dfrac{x-y+1}{y} = -\dfrac{1}{a+b}$.

Divide equation (a) by this, and we get

$$\frac{x}{y} = -\frac{a-b}{a+b},$$

and the solution is comparatively simple.

2. Simplify

$$\frac{a^2-bc}{(a-b)(a-c)} + \frac{b^2+ac}{(b+c)(b-a)} + \frac{c^2+ab}{(c-a)(c+b)}.$$

By properly arranging the signs before these fractions, the whole may be considerably simplified; thus,

$$\frac{a^2-bc}{(a-b)(a-c)} - \frac{b^2+ac}{(a-b)(b+c)} - \frac{c^2+ab}{(a-c)(b+c)}.$$

One quantity like $b-a$ may be changed in either numerator or denominator, and written $a-b$ if the sign in front of the fraction is changed. If, however, the signs of two distinct sets of quantities are changed, no change takes place in the sign before the fraction, whether the quantities are both in the same part of the fraction or one in each.

(b) Reduction and Manipulation of Algebraic Fractions.

In regard to this part of the section, we may add that the examples given are usually dependent on a knowledge of factors, and great experience is necessary before any self-taught student can become completely master of factors.

The following examples in fractions are worth study:—

1. Simplify

$$\frac{1}{1+\dfrac{1}{x+\dfrac{1}{x+1}}}.$$

Fractions like these should be worked upwards from the lowest part of the denominator.

$$\text{Thus}\quad \frac{1}{1+\dfrac{1}{x+\dfrac{1}{x+1}}} = \frac{1}{1+\dfrac{1}{\dfrac{x(x+1)+1}{x+1}}}$$

$$= \frac{1}{1+\dfrac{x+1}{x(x+1)+1}}$$

$$=\frac{1}{\dfrac{\{x(x+1)+1\}+x+1}{x(x+1)+1}}=\frac{x(x+1)+1}{\{x(x+1)+1\}+x+1}$$

$$=\frac{x^2+x+1}{x^2+x+1+x+1}$$

$$=\frac{x^2+x+1}{x^2+2x+2}.$$

2. Simplify
$$\frac{x}{1-\dfrac{x}{1+x+\dfrac{x}{1-x+x^2}}}$$

This
$$=\frac{x}{1-\dfrac{x}{\dfrac{1+x^3+x}{1-x+x^2}}}=\frac{x}{1-\dfrac{x(1-x+x^2)}{1+x+x^3}}$$

$$=\frac{x}{\dfrac{(1+x+x^3)-x(1-x+x^2)}{1+x+x^3}}$$

$$=\frac{x(1+x+x^3)}{(1+x+x^3)-x(1-x+x^2)}$$

$$=\frac{x(1+x+x^3)}{1+x+x^3-x+x^2-x^3}$$

$$=\frac{x(1+x+x^3)}{1+x^2}.$$

(c) Ratio, Proportion, and Variation.

Ratio is defined as the relation which one quantity bears to another with respect to magnitude, the comparison being made by considering what multiple, part, or parts the first quantity is of the second.

The ratio of $a:b$ is $\frac{a}{b}$.

The following are the chief points in ratio to which the student must give his attention :—

1. The ratio of a to b = that of $ma : mb$.

2. The meaning of equality, greater inequality, and less inequality.

3. The effect of adding a small quantity to each term in a ratio of greater inequality and also of less inequality.

4. The meaning of the terms duplicate, triplicate, subduplicate, subtriplicate, etc., as applied to ratio.

5. The ratio of $a^n : (a \pm x)^n =$ ratio of $a : a \pm nx$ nearly, whether n be integral or fractional.

6. If $a : b :: b : c :: c : d$, the ratio compounded of these ratios is $a : d$—that is, the ratio of the first antecedent to the last consequent.

7. If the ratios of $a : b$, $c : d$, $e : f$ are all equal, then each of the ratios $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} = \left(\frac{pa^n + qc^n + re^n}{pb^n + qd^n + rf^n}\right)^{\frac{1}{n}}$ where p, q, r, n have any values whatever.

Proportion is the equality of two ratios. Thus, if there are two ratios, $a : b$ and $c : d$, and these ratios are equal, then the ratios $\frac{a}{b} = \frac{c}{d}$ form a proportion, and $a : b :: c : d$. If $a : b :: c : d$, the following are the most important points for the student to remember:—

1. $ad = bc$, or the product of extremes = product of means.

2. If any three terms of a proportion be given, the fourth may be found; thus,

$$a = \frac{bc}{d},\ b = \frac{ad}{c},\ c = \frac{ad}{b},\ d = \frac{bc}{a}.$$

3. If $a : b :: c : d$, and $c : d :: e : f$, then $a : b :: e : f$.
4. If $a : b :: c : d$, then $b : a :: d : c$.
5. If $a : b :: c : d$, then $a : c :: b : d$.
6. If $a : b :: c : d$, then $a \pm b : b :: c \pm d : d$.
7. If $a : b :: c : d$, then $a \pm b : a \mp b :: c \pm d : c \mp d$.
8. If $a : b :: c : d :: e : f$, etc.,
then $a : b :: a + c + e$, etc. $: b + d + f$, etc.
9. If $a : b :: c : d$, then $ma : mb :: \frac{c}{n} : \frac{d}{n}$.
10. If $a : b :: c : d$, then $ma : \frac{b}{n} :: mc : \frac{d}{n}$.
11. If $a : b :: c : d :: e : f :: g : h$, then $ae : bf :: cg : dh$.
12. If $a : b :: c : d$, then $a^n : b^n :: c^n : d^n$, whether n be integral or fractional.
13. If $a : b :: c : d$,
then $ma \pm nb : pa \pm qb :: mc \pm nd : pc \pm qd$.
14. If two numbers, a and b, be prime to each other, they are the least in the proportion.
15. If $a : b :: b : c$, then $ac = b^2$.

16. If $a : b :: b : c$, then $a : c :: a^2 : b^2$.

17. If $a : b :: b : c :: c : d$, then $a : d :: a^3 : b^3$.

Wood's *Algebra* by Lund is the best for studying proportion.

When the proportion $a : b :: c : d$ is given to prove a certain equality, care must be taken to begin with the given proportion, and with it build up the equality given. On the other hand, when an equality is given to prove a proportion, the equality must be simplified until some form of the proportion is obtained.

We now proceed to work out a few of the proportion examples which have been given.

Ex. 1. If $(a+b+c+d)(a-b-c)+d=(a+b-c-d)(a-b+c-d)$, show that a, b, c, d are proportionals.

Arranging the two quantities on each side of the equation as the sum and difference of two quantities, we have

$$\{(a+d)+(b+c)\}\{(a+d)-(b+c)\}$$
$$=\{(a-d)+(b-c)\}\{(a-d)-(b-c)\};$$
$$\therefore (a+d)^2-(b+c)^2=(a-d)^2-(b-c)^2;$$
$$\therefore a^2+2ad+d^2-b^2-2bc-c^2=a^2-2ad+d^2-b^2+2bc-c^2.$$

Cancelling common quantities,

$$2ad-2bc=2bc-2ad;$$
$$\therefore 4ad=4bc;$$
$$\therefore ad=bc; \therefore a : b :: c : d.$$

Ex. 2. If $a : b :: c : d$, show that

$$ma^2+nb^2 : ma^2-nb^2 :: mc^2+nd^2 : mc^2-na^2.$$

If $a : b :: c : d$, then $\frac{a}{b}=\frac{c}{d}$;

$$\therefore \frac{a^2}{b^2}=\frac{c^2}{d^2};$$
$$\therefore \frac{ma^2}{nb^2}=\frac{mc^2}{nd^2};$$
$$\therefore ma^2+nb^2 : ma^2-nb^2 :: mc^2+nd^2 : mc^2-nd^2.$$

The reduction of equations is often facilitated by the application of proportion.

Ex. 3. Find x from the equation,

$$\frac{\sqrt{x+a}+\sqrt{x}}{\sqrt{x+a}-\sqrt{x}}=c.$$

This resembles a proportion $a : b :: c : d$, where

$$a=\sqrt{x+a}+\sqrt{x},\ b=\sqrt{x+a}-\sqrt{x},$$
$$c=c, \qquad d=1.$$

Now, if $a : b :: c : d$, $a+b : a-b :: c+d : c-d$;

$$\therefore 2\sqrt{x+a} : 2\sqrt{x} :: c+1 : c-1;$$

$$\therefore \sqrt{x+a} : \sqrt{x} :: c+1 : c-1;$$

$$\therefore \frac{\sqrt{x+a}}{\sqrt{x}} = \frac{c+1}{c-1};$$

$$\therefore \frac{x+a}{x} = \left(\frac{c+1}{c-1}\right)^2;$$

$$\therefore 1+\frac{a}{x} = \left(\frac{c+1}{c-1}\right)^2;$$

$$\therefore \frac{a}{x} = \left(\frac{c+1}{c-1}\right)^2 - 1 = \frac{(c+1)^2-(c-1)^2}{(c-1)^2}$$

$$= \frac{(c^2+2c+1)-(c^2-2c+1)}{(c-1)^2};$$

$$\therefore \frac{a}{x} = \frac{4c}{(c-1)^2};$$

$$\therefore \frac{x}{a} = \frac{(c-1)^2}{4c},$$

and $x = \dfrac{a(c-1)^2}{4c}$.

Variation should be read in Todhunter's *Algebra for Colleges and Schools*. All the propositions in chapter xxviii. should be carefully mastered.

The following are the chief definitions and formulæ in variation :—

If m is constant,

1. If $A = mB$, A is said to vary as B.

2. If $A = \dfrac{m}{B}$, A is said to vary inversely as B.

3. If $A = mBC$, A is said to vary jointly as B and C.

4. If $A = m\dfrac{B}{C}$, A is said to vary directly as B, and inversely as C.

5. If $A \propto B$ and $B \propto C$, then $A \propto C$.

6. If $A \propto C$ and $B \propto C$, then $A \pm B \propto C$, and $\sqrt{AB} \propto C$.

7. If $A \propto BC$, then $B \propto \dfrac{A}{C}$ and $C \propto \dfrac{A}{B}$.

8. If $A \propto B$ and $C \propto D$, then $AC \propto BD$.

9. If $A \propto B$, then $A^n \propto B^n$.

10. If $A \propto B$ when C is invariable, and $A \propto C$ when

B is invariable, then A will vary as BC when both B and C are variable.

We conclude with a few examples :—

Ex. 1. If x, y, z are variable, but their sum constant, and if $(x-y+z)(x+y-z)$ varies as yz, prove that yz varies as $(y+z-x)$. (1875.)

Let $x+y+z=m$.

$$\text{And } (x-y+z)(x+y-z)=nyz\,;$$
$$\therefore x^2-(y-z)^2=nyz.$$

Adding $4yz$ to each side,

$$x^2-(y+z)^2=yz\,(n+4)\,;$$
$$\therefore (x+y+z)(x-y-z)=(n+4)\,yz.$$

But $x+y+z=m$;

$$\therefore m\,(x-y-z)=(n+4)\,yz\,;$$
$$\therefore -m\,(y+z-x)=(n+4)\,yz\,;$$
$$\therefore yz=(y+z-x)\left(-\frac{m}{n+4}\right),$$

which varies as $y+z-x$.

Ex. 2. Two spheres of metal of radii r, r', are melted so as to make one sphere. Find the radius of the sphere produced.

The vol. of a sphere $\propto$ as cube of radius; $\therefore$ the vol. of the spheres will be mr^3, mr'^3 respectively;

$$\therefore \text{sum of vols.} = m\,(r^3+r'^3).$$

Let R be the radius of the sphere produced,

$$\text{then its vol.} = m\text{R}^3\,;$$
$$\therefore m\text{R}^3 = m\,(r^3+r'^3)\,;$$
$$\therefore \text{R}=\sqrt[3]{(r^3+r'^3)}.$$

(*d*) PERMUTATIONS AND COMBINATIONS.

There is some ambiguity in the terms used by different writers on these subjects.

The different order in which any quantities can be arranged is called their Permutations. Thus, *a*, *b*, *c*, taken two together, may be arranged *ab*, *ba*, *ac*, *ca*, *bc*, *cb*. These are the permutations of *a*, *b*, *c*, taken two together. If taken three together, their permutations are *abc*, *bca*, *acb*, *bac*, *cba*, *cab*.

Colenso, however, and some other writers, apply the term permutation to quantities taken all together *only;* and when less than the total number of the quantities are taken together, they are called 'variations.' Thus,

when a, b, c are taken two at a time, they would call them the 'variations of a, b, c, taken two at a time;' but when taken all together, they would say, 'permutations of a, b, c.'

Combinations are the different collections which can be made with the same quantities without regarding the order in which they are placed. Thus, with two quantities, a and b, we may make two permutations, ab, ba; these, however, make only one combination.

If we have three quantities, a, b, c, we may have six permutations, as shown above, but there are only three combinations, viz. ab, ac, bc, for the combinations of ab and ba, of ac and ca, of bc and cb, are the same.

The number of permutations taken two at a time $= n\,(n-1)$.

Taken three at a time $= n\,(n-1)\,(n-2)$.

Taken four at a time $= n\,(n-1)\,(n-2)\,(n-3)$.

Taken r at a time $= n\,(n-1)\,(n-2)\ldots(n-r+1)$.

The proof of this formula should be well studied in chap. xxxiv. of Todhunter's *Algebra for Colleges and Schools.*

Taken all together, $n=r$, and the number of permutations

$$\begin{aligned} &= n\,(n-1)\,(n-2)\;.\;.\;.\;(n-n+1)\\ &= n\,(n-1)\,(n-2)\;.\;.\;.\;1\\ &= \text{product of } n \text{ consecutive numbers}\\ &\quad\ \text{beginning with unity.}\end{aligned}$$

The expression $n\,(n-1)\,(n-2)\;.\;.\;.\;1$ is frequently written $\underline{|n}$, and means all the numbers from unity to n multiplied together, and is usually designated as 'factorial n.'

Similarly, $\underline{|n-r}$ means the product of all numbers from unity to $n-r$.

The number of *combinations* of n things, taken r at a time $= \dfrac{n\,(n-1)\,(n-2)\ldots(n-r+1)}{\underline{|r}}$; and taken $n-r$ at a time, the number of combinations will be

$$\frac{n\,(n-1)\,(n-2)\ldots\{n-(n-r)+1\}}{\underline{|n-r}}$$

$$= \frac{n\,(n-1)\,(n-2)\ldots(r+1)}{\underline{|n-r}}.$$

Multiplying numerator and denominator by $\underline{|r}$, the number of combinations of n things taken $n-r$ together

$$=\frac{\{n(n-1)(n-2)\ldots(r+1)\}\,\underline{|r}}{\underline{|r}\,\underline{|n-r}}$$

$$=\frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}.$$

The formula for n things taken r at a time is (as above)

$$\frac{n(n-1)(n-2)\ldots(n-r+1)}{\underline{|r}}.$$

And if we multiply this by $\underline{|n-r}$, we have

$$\frac{n(n-1)(n-2)\ldots(n-r+1)\,\underline{|n-r}}{\underline{|r}\,\underline{|n-r}}$$

$$=\frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}};$$

$\therefore$ the combinations of n things taken r at a time = combinations of n things taken $n-r$ at a time.

For example, if we have 6 letters, the combinations of these taken 4 at a time will be the same as those of 6 taken 2 at a time.

Combinations of 6 things taken 4 at a time

$$=\frac{\underline{|6}}{\underline{|4}\,\underline{|6-4}}=\frac{1.2.3.4.5.6}{1.2.3.4.1.2}=15.$$

Combinations taken 2 at a time

$$=\frac{\underline{|6}}{\underline{|2}\,\underline{|6-2}}=\frac{1.2.3.4.5.6}{1.2.1.2.3.4}=15.$$

It will be observed that the number of combinations which can be formed with n things varies with the value of r, and the following facts will enable us to find the value of r, which will give the greatest number of combinations.

The number of combinations will go on increasing as r increases, until $\frac{n-r+1}{r}$ first becomes equal to or less than unity.

If r be such that $\frac{n-r+1}{r}=1$,—that is, if $r=\frac{1}{2}(n+1)$, —and if n be odd, so that $\frac{1}{2}(n+1)$ be a whole number, then this value of r will give the greatest number of

combinations; but if n be even, $\frac{n}{2}$ will be the value of r, which will give the greatest number of combinations.

If there are n letters, p of them being A's, q of them B's, and r of them C's, and all the rest unlike, the permutations of these letters will be

$$=\frac{\lfloor n}{\lfloor p \lfloor q \lfloor r}.$$

Ex. 1. The number of permutations that can be formed out of the letters in the word Mississippi.

$$n=11;$$
i occurs 4 times;
s occurs 4 times;
p occurs 2 times,

$$\therefore \text{permutations}=\frac{\lfloor 11}{\lfloor 4 \lfloor 4 \lfloor 2}$$
$$=\frac{1.2.3.4.5.6.7.8.9.10.11}{1.2.3.4.1.2.3.4.1.2}$$
$$=\frac{5.6.7.8.9.10.11}{1.2.3.4.1.2}$$
$$=5.7.9.10.11=34{,}650.$$

The number of combinations of two sets of things containing respectively p and q things, m being taken out of one set, and n out of the other,

$$=\frac{p(p-1)(p-2)\dots(p-m+1)}{1.2.3\dots m}$$
$$\times\frac{q(q-1)\dots(q-m+1)}{1.2\dots n}.$$

Ex. 2. If we have 10 black and 3 white balls, how many sets containing four black and two white ones may be made?

Here $p=10$, $q=3$, $m=4$, $n=2$;

$$\therefore \text{number required}=\frac{10.9.8.7}{1.2.3.4}\times\frac{3\times 2}{1\times 2}$$
$$=630.$$

The whole number of permutations of n things when each may occur once, twice, etc. to r times $=n^r$.

Ex. 3. In how many ways can I select two white balls and three red ones out of an urn containing seven white balls and ten red? (1873, 1st B.A.)

Referring this to the last formula but one, we have $p=7$, $q=10$, $m=2$, $n=3$;

$\therefore$ number of ways required

$$=\frac{7.6}{1.2}\times\frac{10.9.8.7.6.5.4}{1.2.3}.$$
$$=7.3.10.9.8.7.5.4$$
$$=210\times72\times140$$
$$=2,116,800.$$

Ex. 4. Three persons have four coats, five vests, and six hats between them. In how many different ways can they dress themselves?

When dressed, three articles of each sort will be in use. This, then, will be an example of the same formula as the last example, in which $p=4$, $q=5$, $r=6$, $m=3$, $n=3$, $s=3$;

$$\therefore \text{ number of ways}=\frac{4.3.2}{1.2.3}\times\frac{5.4.3}{1.2.3}\times\frac{6.5.4}{1.2.3}$$
$$=4\times10\times20=800.$$

The student will do well to study the examples given in the book above referred to.

(*e*) 1. Arithmetical Progression.

The usual formulæ in this subject present no difficulty in their proofs, so that we only enunciate them for reference.

Let a = first term,
d = common difference,
l = last term,
n = number of terms,
S = sum of terms.

Then the series is

$$a+(a+d)+(a+2d)+\text{etc.}\ldots a+\overline{n-1}d;$$
$$\therefore l=a+\overline{n-1.d},$$
$$S=\left\{2a+\overline{n-1}.d\right\}\frac{n}{2}$$
$$=\left\{a+a+\overline{n-1}.d\right\}\frac{n}{2}=(a+l)\frac{n}{2},$$
$$d=\frac{n-1}{l-a}.$$

n can be found by means of a quadratic only, except

the sum and the first and last terms are given, and in that case $n=\frac{2S}{a+l}$.

It will be noticed that a single mean between two terms a and b will be $\frac{a+b}{2}$. Thus, let x be the mean, and a and b the two terms, one before, and one after the mean.

$x-a$ and $b-x$ will each be the difference between two terms;

$$\therefore x-a=b-x;\ \because\ 2x=a+b;$$

$$\therefore x=\frac{a+b}{2}.$$

To find m means between two quantities a, b, we find the common difference—

$$\text{viz. } d=\frac{l-a}{n-1}.$$

But if we have n terms, we have $n-2$ means;

$$\therefore m=n-2;\ \therefore m+1=n-1.$$

And if m be the number of means, $d=\frac{l-a}{m+1}$, which gives the common difference at once.

The common difference may also be found when n, S, and a are given.

$$\text{Thus } S=\left\{2a+\overline{n-1}\,.\,d\right\}\frac{n}{2};$$

$$\therefore \frac{2S}{n}=2a+\overline{n-1}\,.\,d;$$

$$\therefore \frac{2S}{n}-2a=d\,.\,(n-1);$$

$$\therefore \frac{2\,(S-na)}{n}=d\,(n-1);$$

$$\therefore d=\frac{2\,(S-an)}{n\,(n-1)}.$$

The sum of any two terms equidistant from the beginning and end is equal to the sum of the first and last terms.

If the number of terms is odd, the middle term is half the sum of the first and last terms.

If the number of terms is even, the sum of the two terms in the middle of the series will be equal to the sum of the first and last terms.

Let M be the middle term, and let n, the number of terms, be odd.

$$\begin{aligned}
\text{Then } M &= \tfrac{1}{2}(a+l) \\
&= \tfrac{1}{2}(a+a+\overline{n-1}\,.\,d) \\
&= \tfrac{1}{2}(2a+(n-1)\,d);
\end{aligned}$$

$$\therefore\ 2M = 2a+(n-1)\,d\,;$$

$$\begin{aligned}
\therefore\ 2M \times \frac{n}{2} &= \left(2a+\overline{n-1}\,.\,d\right)\frac{n}{2} \\
&= S;
\end{aligned}$$

$\therefore$ $S = Mn$ = middle term $\times$ number of terms when n is odd.

Let n be even, and M', M'' the two middle terms.

$$\begin{aligned}
\text{Then } M'+M'' &= a+l \\
&= 2a+\overline{n-1}\,.\,d\,;
\end{aligned}$$

$$\begin{aligned}
\therefore\ \frac{n}{2}(M'+M'') &= \left(2a+\overline{n-1}\,.\,d\right)\frac{n}{2} \\
&= S,
\end{aligned}$$

or the sum = half the sum of the middle terms multiplied by the number of terms.

Thus the sum of the series $\frac{2}{5}+\frac{7}{15}+\frac{4}{15}+\frac{1}{15}$ to 7 terms would be $7 \times$ fourth term $= 7 \times \frac{1}{15} = \frac{7}{15}$.

And to 8 terms, the sum of this series would be

$$\left(\tfrac{1}{15} + \text{next term}\right) \times \tfrac{8}{2}$$
$$= \left(\tfrac{1}{15} - \tfrac{2}{15}\right) \times 4 = -\tfrac{4}{15}.$$

Sometimes arithmetical series are expressed in a general form; thus we sometimes read of the series $2n+1$, $2n-3$, $3n-4$, and so on. The series themselves may be found from these general forms by substituting successively the numbers 1, 2, 3, etc. for n.

Thus the series $2n+1$ is 3, 5, 7, 9, etc.

" " $2n-3$ is -1, $+1$, 3, 5, etc.

" " $3n-4$ is -1, 2, 5, 8, etc.

The expressions here used are only expressions for the general term, and in finding the sum, they may be used for the last term.

Thus the sum of the series represented by $2n+1$ would be, for n terms,

$$\left(a+2n+1\right)\frac{n}{2}. \quad \text{But } a=3\,;$$

$$\therefore\ S = \left(3+2n+1\right)\frac{n}{2} = n\,(n+2).$$

If we wish to find the sum of, say, 4 terms, we put $n=4$;

$$\therefore \text{ sum of 4 terms} = 4\ (4+2) = 24,$$
$$\text{and sum of 10 terms} = 10\ (10+2) = 120.$$

We conclude with a few examples :—

Ex. 1. Find the number of means between 1 and 29 when the second : the last but two :: 3 : 7.

Let d be the common difference.

$$\text{Then } 1+2d : 29-2d :: 3 : 7;$$
$$\therefore 87-6d = 7+14d;$$
$$\therefore 80 = 20d,$$
$$d = 4.$$

$$\text{But } d = \frac{l-a}{m+1}, \text{ or } 4 = \frac{29-1}{m+1}, \ 4 = \frac{28}{m+1},$$
$$4m+4 = 28; \ \therefore m = 6.$$

Ex. 2. Sum to $2n+1$ terms the series whose general term is $6-\frac{m}{2}$.

$$\text{If } m=1, \text{ first term} = 5\tfrac{1}{2}.$$
$$\text{If } m=2, \text{ second term} = 5;$$
$$\therefore d = -\tfrac{1}{2};$$
$$\therefore S = \left\{ 2\times 5\tfrac{1}{2} + (2n+1-1)\left(-\tfrac{1}{2}\right) \right\} \frac{2n+1}{2}$$
$$= \left\{ 11-n \right\} \frac{2n+1}{2} = \frac{(1+2n)(11-n)}{2}.$$

Ex. 3. If the first term is 27, the fourth term 18, and the sum 117, find the number of terms and the last term. (B.A., 1875.)

$$a = 27.$$
$$a+3d = 18; \ \therefore 27+3d = 18; \ \therefore d = -3.$$
$$S = 117; \ \therefore 117 = \left\{ 2a + \overline{n-1}\,.\,d \right\} \frac{n}{2}$$
$$= \left\{ 54 + (n-1)(-3) \right\} \frac{n}{2}$$
$$= \left\{ 57-3n \right\} \frac{n}{2};$$
$$\therefore 234 = 57n - 3n^2;$$
$$\therefore 78 = 19n - n^2;$$
$$\therefore n^2 - 19n + 78 = 0,$$
$$\text{or } (n-6))\ n-13) = 0;$$

$\therefore n = 6$ or 13, each of which answer the requirements.

$\therefore$ The last term

$$\begin{aligned} &= a+\overline{n-1}\,.\,d \\ &= 27+(6-1)(-3)=27-18+3 \\ &\qquad\qquad\qquad\qquad\;\; =12, \end{aligned}$$

or if $n=13$, $l=27+(13-1)(-3)=27-39+3$
$$=-9.$$

Ex. 4. If the sum of an A.P. $=\frac{n}{2}\left(\frac{n^2}{3}+\frac{1}{4}\right)$ for n terms, find the eighth term.

$$\text{Let } n=1;$$
$$\therefore \text{ first} = \tfrac{1}{2}\left(\tfrac{1}{3}+\tfrac{1}{4}\right) \tfrac{7}{24}.$$
$$\text{Sum of 2 terms} = \tfrac{2}{2}\left(\tfrac{4}{3}+\tfrac{1}{4}\right)=\tfrac{19}{12};$$
$$\therefore \text{ second term} = \tfrac{19}{12}-\tfrac{7}{24}=\tfrac{31}{24};$$
$$\therefore \text{ common difference} = \tfrac{31}{24}-\tfrac{7}{24}=1;$$
$$\therefore \text{ eighth term} = a+7d=\tfrac{7}{24}+7=7\tfrac{7}{24}.$$

Ex. 5. There are m arithmetic progressions, each beginning with 1, and their common differences are 1, 2, 3, etc., m. Show that the sum of their nth terms
$$=\tfrac{1}{2}\{m^2(n-1)+m(n+1)\}.$$

1st series is 1, 2, 3, 4, etc., and nth term is n,
2d " " 1, 3, 5, 7, etc., " " $2n-1$,
3d " " 1, 4, 6, 8, etc., " " $3n-2$;
$\therefore$ in summing their mth terms, $a=n$, $d=n-1$;

$$\begin{aligned} \therefore S &= \left\{2n+(m-1)(n-1)\right\}\frac{m}{2} \\ &= \left\{2n+mn-m-n+1\right\}\frac{m}{2} \\ &= \left\{n+mn-m+1\right\}\frac{m}{2} \\ &= \left\{m(n-1)+(n+1)\right\}\frac{m}{2} \\ &= \tfrac{1}{2}\{m^2(n-1)+m(n+1)\}. \end{aligned}$$

Ex. 6. The sum of five numbers in A.P. is 30, and the sum of their squares 220; find the numbers.

As there is an odd number of terms in the series, take x as the middle term, and y as the common difference. Then the series will be

$$x-2y,\ x-y,\ x,\ x+y,\ x+2y.$$
$$\text{Then } x-2y+x-y+x+x+y+x+2y=30;$$
$$\therefore 5x=30;\ \therefore x=6.$$

Again,
$$(x-2y)^2+(x-y)^2+x^2+(x+y)^2+(x+2y)^2=220.$$

Squaring and adding,

$$5x^2+10y^2=220;$$
$$\therefore x^2+2y^2=44.$$
$$\text{But } x=6;$$
$$\therefore 36+2y^2=44;$$
$$\therefore 2y^2=8,$$
$$y^2=4,$$
$$y=\pm 2;$$

$\therefore$ the series is 2, 4, 6, 8, 10.

Ex. 7. There are four numbers in A.P., the sum of the squares of the extremes $=101$, and of the means 65; find them.

As the number of terms is even, take $2y$ as the common difference, and let the series be $x-3y$, $x-y$, $x+y$, $x+3y$.

$$\text{Then } (x-3y)^2+(x+3y)^2=101;$$
$$\therefore 2x^2+18y^2=101 \quad . \quad . \quad . \quad (1).$$
$$\text{Again, } (x-y)^2+(x+y)^2=65;$$
$$\therefore 2x^2+2y^2=65 \quad . \quad . \quad . \quad (2).$$

Subtracting (2) from (1)—

$$16y^2=36,$$
$$4y^2=9,$$
$$y=\pm\tfrac{3}{2},$$
$$\text{and } x^2=\tfrac{121}{4};\ \therefore x=\pm\tfrac{11}{2}.$$

By arranging in this way, we simplify the solution considerably.

---

2. Geometrical Progression.

Quantities are in geometrical progression when each term is equal to the one preceding it, multiplied by a constant quantity. This constant quantity is called the common ratio, and may be greater or less than unity, positive or negative.

As geometrical progression is included in the Matriculation Syllabus, the proofs of the formulæ will be familiar to the reader, and it will only be necessary to repeat them for reference, and to work out a few examples of a more advanced character than those already given.

The chief formulæ are as follows :—

S being the sum to n terms,
Σ the sum to infinity,
a the first term,
l the last term,
n the number of terms.

(1) $l = ar^{n-1}$.

(2) $S = \frac{a(r^n - 1)}{r - 1} = \frac{a(1 - r^n)}{1 - r}$ when r is less than 1.

(3) $= \frac{rl - a}{r - 1}$.

(4) $\Sigma = \frac{a}{1 - r}$.

And if m be the number of means,

(5) $r = \sqrt[m+1]{\frac{l}{a}}$. Also, (6) $b^2 = ac$.

The formulæ relating to the value of recurring decimals will be found under the head of Decimals.

The difference between the sum of an infinite series taken only to n terms, and the sum to infinity, must be read in Todhunter's *Algebra for Colleges and Schools*, sections 465, 466.

We may always find the common ratio by dividing one term by the next preceding it.

Ex. 1. Sum to n terms the series $2 - 2^2 + 2^3 - 2^4 +$ etc.

In this case, $r = \frac{-2^2}{2} = -2$;

$\therefore S = \frac{a(r^n - 1)}{r - 1} = \frac{2\{(-2)^n - 1\}}{-2 - 1} = -\frac{2}{3}\{(-2)^n - 1\}$,

which will be positive or negative according as n is odd or even.

Ex. 2. Sum to infinity $\frac{\sqrt{2}+1}{\sqrt{2}-1} + \frac{1}{2-\sqrt{2}} + \frac{1}{2}$.

$$r = \frac{1}{2-\sqrt{2}} \div \frac{\sqrt{2}+1}{\sqrt{2}-1} = \frac{1}{\sqrt{2}(\sqrt{2}-1)} \times \frac{\sqrt{2}-1}{\sqrt{2}+1}$$

$$= \frac{1}{\sqrt{2}(\sqrt{2}+1)};$$

$$\therefore \Sigma = \frac{\sqrt{2}+1}{\sqrt{2}-1} \div \left(1 - \frac{1}{\sqrt{2}(\sqrt{2}+1)}\right)$$

$$= \frac{\sqrt{2}+1}{\sqrt{2}-1} \div \left(\frac{2+\sqrt{2}-1}{2+\sqrt{2}}\right)$$

$$= \frac{\sqrt{2}+1}{\sqrt{2}-1} \times \frac{2+\sqrt{2}}{\sqrt{2}+1}$$

$$= \frac{2\sqrt{2}+2+2+\sqrt{2}}{2-1}$$

$$= 3\sqrt{2}+4.$$

Ex. 3. Sum to infinity $\frac{1}{5}+\frac{3}{5^2}+\frac{1}{5^3}+\frac{3}{5^4}+\text{etc.}$

$$=\left(\frac{1}{5}+\frac{3}{5^2}\right)+\frac{1}{5^2}\left(\frac{1}{5}+\frac{3}{5^2}\right)+\text{etc.}$$

Take $\frac{1}{5}+\frac{3}{5^2}$ as first term, $\frac{1}{5^2}=r$;

$$\therefore\ \Sigma=\frac{\frac{1}{5}+\frac{3}{5^2}}{1-\frac{1}{5^2}}=\frac{25}{24}\left(\frac{1}{5}+\frac{3}{5^2}\right)$$

$$=\frac{25}{24}\left(\frac{8}{25}\right)=\frac{8}{24}=\frac{1}{3}.$$

Ex. 4. What is the error in taking the sum to infinity as the sum of one thousand terms of the series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\text{etc.}$?

$$\text{Sum to infinity}=\frac{a}{1-r}.$$

$$\text{Sum to } n \text{ terms}=\frac{a(1-r^n)}{1-r}=\frac{a}{1-r}-\frac{ar^n}{1-r};$$

$$\therefore \text{ the error will be } \frac{a}{1-r}-\frac{ar^n}{1-r}-\frac{a}{1-r}=-\frac{ar^n}{1-r};$$

$\therefore$ the sum to infinity will be too great by $\frac{ar^n}{1-r}$.

Now when $n=1000$ and $r=\frac{1}{2}$,

$$\text{error}=\frac{\frac{1}{2}\cdot\frac{1}{2^{1000}}}{1-\frac{1}{2}}=\frac{1}{2^{1000}}.$$

Ex. 5. If an arithmetic mean between a and b be twice as great as the geometric mean, find the ratio between a and b.

$$\text{Arithmetic mean}=\frac{a+b}{2}.$$

$$\text{Geometric mean}=\sqrt{ab};$$

$$\therefore\ \frac{a+b}{2}=2\sqrt{ab}\ (1);\ \therefore\ a+b=4\sqrt{ab}\ (2);$$

$$\therefore\ a^2+2ab+b^2=16ab;$$

$$\therefore\ a^2-2ab+b^2=12ab;$$

$$\therefore\ a-b=2\sqrt{3}\,\sqrt{ab};$$

$$\therefore\ \frac{a-b}{\sqrt{ab}}=2\sqrt{3}\ (3),\ \text{or}\ \frac{a}{\sqrt{ab}}-\frac{b}{\sqrt{ab}}=2\sqrt{3},$$

$$\text{or}\ \frac{\sqrt{a}}{\sqrt{b}}-\frac{\sqrt{b}}{\sqrt{a}}=2\sqrt{3}.$$

But by (2), in the same way, $\frac{\sqrt{a}}{\sqrt{b}}+\frac{\sqrt{b}}{\sqrt{a}}=4.$

Adding these results together,

$$2\frac{\sqrt{a}}{\sqrt{b}}=2\sqrt{3}+4;$$

$$\therefore\ \frac{\sqrt{a}}{\sqrt{b}}=\sqrt{3}+2.$$

Subtracting the same quantities, $2\frac{\sqrt{b}}{\sqrt{a}}=4-2\sqrt{3};$

$$\therefore\ \frac{\sqrt{b}}{\sqrt{a}}=2-\sqrt{3}.$$

Multiplying this after inverting with the preceding,

$$\frac{a}{b}=\frac{2+\sqrt{3}}{2-\sqrt{3}}.$$

Ex. 6. Find the sum of the series,

$$x^n+x^{n-1}y+x^{n-2}y^2+\text{etc.}\ .\ .\ .\ xy^{n-1}+y^n.$$

Here $a=x^n$, $r=\frac{y}{x}$;

$$\therefore\ \Sigma=\frac{x^n}{1-\frac{y}{x}}=\frac{x^{n+1}}{x-y}=x^{n+1}\,(x-y)^{-1}.$$

Ex. 7. If $S=1+R+R^2+R^3+\text{etc.}$ to infinity, and $s=1+r+r^2+r^3+\text{etc.}$ to infinity, find the sum of $1+Rr+R^2r^2+R^3r^2+\text{etc.}$ to infinity.

Multiplying the given series together, we get

$$\begin{aligned} Ss&=1+R+R^2+R^3+\text{etc.}+r+Rr+R^2r+R^3r\\ &\quad+\text{etc.}+r^2+Rr^2+R^2r^2+R^3r^2+\text{etc.}+r^3\\ &\quad+Rr^3+R^2r^3+R^3r^3+\text{etc.}\\ &=(1+Rr+R^2r^2+\text{etc.})\,(1+R+R^2+\text{etc.})\\ &\quad+(r+r^2+\text{etc.})\\ &=(1+Rr+R^2r^2+\text{etc.})\,(1+R+R^2+\text{etc.})\\ &\quad+(1+r+r^2+\text{etc.})-1\\ &=(1+Rr+R^2r^2+\text{etc.})\,(S+s-1); \end{aligned}$$

$\therefore$ sum to infinity of $1+Rr+R^2r^2+$etc.

$$=\frac{Ss}{S+S-1}.$$

We select a few examples from the examination papers.

Ex. 8. Find the sum to infinity of the series,

$$1+\frac{1}{3}+\frac{1}{9}+\text{etc.},$$

and the sum of the least number of terms of the series differing by less than $\frac{1}{1000}$ from the sum to infinity.

$$\Sigma=\frac{a}{1-r};$$

$$\therefore \text{ sum to infinity}=\frac{1}{1-\frac{1}{3}}=\frac{1}{\frac{2}{3}}=\frac{3}{2}.$$

$$S=\frac{a}{1-r}-\frac{ar^n}{1-r}.$$

We have therefore to find n when $\frac{ar^n}{1-r}$ is less than $\frac{1}{1000}$,

$$\text{or when } \frac{1\times\frac{1}{3^n}}{\frac{2}{3}} \text{ is less than } \frac{1}{1000},$$

$$\text{or } \frac{1}{3^n}\times\frac{3}{2}=\frac{1}{2.3^{n-1}} \text{ is less than } \frac{1}{1000},$$

$$\text{or when } \frac{1}{3^{n-1}} \text{ is less than } \frac{1}{500},$$

$$\text{or when } 3^{n-1} \text{ is less than } 500.$$

$$\text{But } 3^5=243,$$

$$\text{and } 3^6=729;$$

$$\therefore n-1=6;\ \therefore n=7;$$

$\therefore$ the sum of 7 terms differs from the sum to infinity by less than $\frac{1}{1000}$ part of unity.

Ex. 9. If S_1, S_2, S_3, etc., S_n be the sum of n geometrical progressions whose first terms are a, $2a$, $3a$ . . . na, show that $S_1+S_2+S_3+$etc. . . . S_n

$$=\frac{n(n+1)}{2}.\left(\frac{r^n-1}{r-1}\right)a,$$

$$S_1=\frac{a\,(r^n-1)}{r-1},$$
$$S_2=2a\,.\,\frac{r^n-1}{r-1},$$
$$S_3=3a\,.\,\frac{r^n-1}{r-1},\text{ etc.},$$
$$\text{and } S_n=na\,.\,\frac{r^n-1}{r-1},$$

$S_1+S_2+S_3+\text{etc.}\ .\ .\ .\ S_n=\frac{r^n-1}{r-1}\,(a+2a+3a+\text{etc.}$ $.\ .\ .\ na)$, and the part in brackets is an arithmetical progression whose sum$=a\,(1+n)\,\frac{n}{2}$;

$$\therefore\ S_1+S_2+S_3+\text{etc.}\ .\ .\ .\ Sn=\frac{r^n-1}{r-1}\cdot\left(\frac{1+n}{2}\right)na,$$

as above.

Ex. 10. Find the sum of

$$\frac{1}{8}+\frac{4}{8^2}+\frac{6}{8^3}+\frac{3}{8^4}+\frac{1}{8^5}+\frac{4}{8^6}+\frac{6}{8^7}+\frac{3}{8^8}$$

to infinity, the numerators 1, 4, 6, 3 recurring (1877).

$$\text{Series}=\left(\frac{1}{8}+\frac{4}{8^2}+\frac{6}{8^3}+\frac{3}{8^4}\right)\left(1+\frac{1}{8^4}+\frac{1}{8^8}+\text{etc.}\right);$$

$$\therefore\ r=\frac{1}{8^4},\ a=\frac{1}{8}+\frac{4}{8^2}+\frac{6}{8^3}+\frac{3}{8^4};$$

$$\therefore\ \Sigma=\frac{\frac{1}{8}+\frac{4}{8^2}+\frac{6}{8^3}+\frac{3}{8^4}}{1-\frac{1}{8^4}}=\frac{8^3+4\times 8^2+6\times 8+3}{8^4-1}$$

$$=\frac{512+256+48+3}{8^4-1}=\frac{819}{64^2-1}=\frac{819}{4095}$$

$$=\frac{91}{455}=\frac{1}{5}.$$

Ex. 11. The first term of a G. P. is 3, and the fourth term $\frac{1}{\sqrt{3}}$. Find its sum to infinity (1869).

$$a=3,$$
$$ar^3=\frac{1}{\sqrt{3}};\ \therefore\ r^3=\frac{1}{3\sqrt{3}};$$
$$\therefore\ r=\frac{1}{\sqrt{3}};$$

$$\therefore \Sigma = \frac{3}{1-\frac{1}{\sqrt{3}}} = \frac{3}{\frac{\sqrt{3}-1}{\sqrt{3}}} = \frac{3\sqrt{3}}{\sqrt{3}-1}.$$

Ex. 12. In a G. P., prove that $r = \frac{S-a}{S-l}$, where S = sum, r ratio, a and l the first and last terms respectively.

$$S = \frac{rl-a}{r-1}; \therefore S(r-1) = rl-a;$$

$$\therefore Sr - S = rl - a, \text{ or } r(S-l) = S-a;$$

$$\therefore r = \frac{S-a}{S-l}.$$

Sometimes series are given which are not composed of quantities in A. P. or G. P., but which may be arranged as such, or their sums found by some artifice.

Ex. 13. Sum the series $1+2x+3x^2+4x^3+$ etc., *ad infin.*, and find its square root.

$$\text{Let } S = 1+2x+3x^2+4x^3+5x^4+\text{etc., } ad\ infin.;$$

$$\therefore Sx = x+2x^2+3x^3+4x^4+\text{etc., } ad\ infin.;$$

$$\text{subtracting, } \therefore S(1-x) = 1+x+x^2+x^3+x^4+\text{etc., } ad\ infin.$$

$$\text{But } 1+x+x^2+x^3+x^4+\text{etc., } ad\ infin. = \frac{1}{1-x};$$

$$\therefore S(1-x) = \frac{1}{1-x};$$

$$\therefore S = \frac{1}{(1-x)^2}, \text{ the sum of the given series;}$$

$$\therefore \sqrt{1+2x+3x^2+4x^3+\text{etc., } ad\ infin.} = \frac{1}{1-x}.$$

Ex. 14. Find the sum of the series 2, 3, 5, 11, 25, etc. to n terms.

This series is

$$2+(4-1)+(8-3)+(16-5)+(32-7)$$
$$= 2+4+8+16+32 \text{ to } n \text{ terms}$$
$$-(1+3+5+7+\text{etc. to } n-1 \text{ terms}).$$

Sum of $2+4+8+16+$ etc. to n terms

$$= \frac{2(2^n-1)}{2-1} = 2(2^n-1).$$

Sum of $1+3+5+$ etc. to $n-1$ terms $=(n-1)^2$;

$\therefore$ sum of 2, 3, 5, 11, 25, etc. $= 2(2^n-1)-(n-1)^2$.

Ex. 15. Sum the series

$$ax+(a+b)x^2+(a+2b)x^3+(a+3b)x^4+\text{etc.}$$

to n terms.

This series may be written

$$ax+ax^2+ax^3+ax^4+\text{etc.}+bx^2+2bx^3+3bx^4+\text{etc.}$$
$$=a\,(x+x^2+x^3+\text{etc. to } n \text{ terms})$$
$$+bx^2\,(1+2x+3x^2+4x^3+\text{etc. to } n-1 \text{ terms}),$$
$$a\,(x+x^2+x^3+\text{etc. to } n \text{ terms})=\frac{a\,.\,x\,.\,(x^n-1)}{x-1}.$$

To sum $1+2x+3x^2+\text{etc.}$ to $n-1$ terms.

Let $S=1+2x+3x^2+4x^3+\text{etc.}\ \ldots\ +(n-1)\,x^{n-2}$,

$$Sx=x+2x^2+3x^3+\text{etc.}+(n-2)\,x^{n-2}$$
$$+(n-1)\,x^{n-1};$$
$$\therefore S\,(1-x)=1+x+x^2+x^3+\text{etc.}\ \ldots$$
$$\ldots\ x^{n-2}-(n-1)\,x^{n-1}$$
$$=\frac{1-x^{n-1}}{1-x}-(n-1)\,x^{n-1}$$
$$=\frac{(n-1)\,x^n-nx^{n-1}+1}{1-x};$$
$$\therefore S=\frac{(n-1)\,x^n-nx^{n-1}+1}{(1-x)^2};$$
$$\therefore bx^2\,(1+2x+3x^3+\text{etc.})$$
$$=\frac{bx^2\,\{(n-1)\,x^n-nx^{n-1}+1\}}{(1-x)^2};$$
$$\therefore ax+(a+b)\,x^2+(a+2b)\,x^3+\text{etc.}$$
$$=ax\,.\,\frac{x^n-1}{x-1}+\frac{bx^2\,\{(n-1)\,x^n-nx^{n-1}+1\}}{(1-x)^2}$$
$$=\frac{ax\,(x^n-1)\,(x-1)+bx^2\,\{(n-1)\,x^n-nx^{n-1}+1\}}{(1-x)^2}.$$

Ex. 16. Sum the series

$$\frac{a-1}{a}\,x+\frac{a-2}{a}\,x^2+\frac{a-3}{a}\,x^3+\text{etc. to } n \text{ terms.}$$

This series $=x+x^2+x^3+x^4+\text{etc. to } n$ terms

$$-\frac{x}{a}\,(1+2x+3x^2+\text{etc.}).$$

Sum of $x+x^2+x^3$ etc. to n terms $=x.\dfrac{1-x^n}{1-x}$.

Sum of $1+2x+3x^2+\text{etc.}$, to n terms

$=\dfrac{nx^{n+1}-(n+1)x^n+1}{(1-x)^2}$ by writing n for $n-1$ in the sum above;

$$\therefore\ \frac{a-1}{a}\,x+\frac{a-2}{a}\,x^2+\frac{a-3}{a}\,x^3+\text{etc. to } n \text{ terms}$$

$$=x.\frac{1-x^n}{1-x}-\frac{x}{a}\cdot\frac{nx^{n+1}-(n+1)x^n+1}{(1-x)^2}$$

$$=\frac{ax(1-x^n)(1-x)-nx^{n+1}-(n+1)x^n+1}{a(1-x)^2}.$$

Ex. 17. Find the sum of the series

$$(S-a)+\{S-(a+ar)\}+\{S-(a+ar+ar^2)\}$$
$$+\{S-(a+ar+ar^2+ar^3)\}+\text{etc.}$$

to n terms and to infinity, when S = the sum of $a+ar+ar^2+$ etc. to n terms.

Sum of $a+ar+ar^2+$ etc. to n terms $=a\,\frac{r^n-1}{r-1}$;

$$\therefore\ S=a\,.\,\frac{r^n-1}{r-1};$$

$$\therefore\ S-a=\frac{a(r^n-1)-a(r-1)}{r-1}=\frac{ar(r^n-1)}{r-1}$$
$$=\frac{ar^n}{r-1}-\frac{ar}{r-1}.$$

$$S-(a+ar)=\frac{a(r^n-1)-a(r^2-1)}{r-1}=\frac{ar^2(r^{n-2}-1)}{r-1}$$
$$=\frac{ar^n}{r-1}-\frac{ar^2}{r-1}.$$

$$S-(a+ar+ar^2)=\frac{a(r^n-1)-a(r^3-1)}{r-1}$$
$$=\frac{ar^3(r^{n-3}-1)}{r-1}=\frac{ar^n}{r-1}-\frac{ar^3}{r-1}.$$

$$S-(a+ar+ar^2+ar^3)=\frac{a(r^n-1)-a(r^4-1)}{r-1}$$
$$=\frac{ar^4(r^{n-4}-1)}{r-1}=\frac{ar^n}{r-1}-\frac{ar^4}{r-1},\text{ etc.};$$

$\therefore$ the sum required = sum of $\frac{ar^n}{r-1}+\frac{ar^n}{r-1}$ + etc. to n terms $-\left(\frac{ar}{r-1}+\frac{ar^2}{r-1}+\frac{ar^3}{r-1}+\text{etc. to } n \text{ terms}\right)$

$$=n\,.\,\frac{ar^n}{r-1}-\text{sum of }\frac{ar}{r-1}+\frac{ar^2}{r-1}+\frac{ar^3}{r-1}+\text{etc.}$$

$$=n\,.\,\frac{ar^n}{r-1}-\frac{a}{r-1}(r+r^2+r^3+\text{etc. to } n \text{ terms})$$

$$=\frac{nar^n}{r-1}-\frac{a}{r-1}\left(r\,.\,\frac{r^n-1}{r-1}\right)$$

$$=\frac{nar^n}{r-1}-\frac{ar(r^n-1)}{(r-1)^2}$$

$$=\frac{nar^n}{r-1}-\frac{ar^{n+1}}{(r-1)^2}+\frac{ar}{(r-1)^2},\text{ which is sum of } n \text{ terms.}$$

Now when the number of terms becomes infinite,

$$\frac{nar^n}{r-1}=\frac{ar^{n+1}}{(r-1)^2};\ \therefore\ \frac{nar^n}{r-1}-\frac{ar^{n+1}}{(r-1)^2}=0;$$

$$\therefore\ \text{sum to infinity}=\frac{ar}{(r-1)^2}.$$

Ex. 18. Find the sum of p terms of the series whose nth term is $na+a^n$.

Let $n=1$; $\therefore$ 1st term $=a+a$.
„ $n=2$; $\therefore$ 2d „ $=2a+a^2$.
„ $n=3$; $\therefore$ 3d „ $=3a+a^3$.
„ $n=4$; $\therefore$ 4th „ $=4a+a^4$.

$\therefore$ sum of the series is $a+2a+3a+4a+$ etc. to p terms, $+a+a^2+a^3+a^4+$ etc. to p terms

$$=(a+pa)\frac{p}{2}+a.\frac{a^p-1}{a-1}$$

$$=\frac{ap}{2}(p+1)+\frac{a(a^p-1)}{a-1}.$$

The formulæ for the sum of $1+2x+3x^2+$ etc. enables us to sum many similar series. The following are examples of its application.

Ex. 19. Sum the series $1+6x+11x^2+16x^3+$ etc. to infinity.

$1+6x+11x^2+16x^3+21x^4+\text{etc.}=1+2x+3x^2+4x^3$ $+$ etc. to n terms, $+4x(1+2x+3x^2+4x^3+\text{etc.})$

$$=(1+4x)(1+2x+3x^2+4x^3+5x^4+\text{etc.})$$

$$=(1+4x)\left(\frac{1}{(1-x)^2}\right)$$

$$=\frac{1+4x}{(1-x)^2}.$$

Ex. 20. Sum the series $1+\frac{2}{3}+\frac{3}{3^2}+\frac{4}{3^3}+\frac{5}{3^4}+$ etc. to 10 terms, to n terms, and to infinity.

This series has the form $1+2x+3x^2+4x^3+$ etc. where $x=\frac{1}{3}$;

$\therefore$ sum to n terms

$$=\frac{nx^{n+1}-(n+1)\,x^n+1}{(1-x)^2} \text{ by Ex. 16}$$

$$=\frac{n.\frac{1}{3^{n+1}}-(n+1)\frac{1}{3^n}+1}{\left(1-\frac{1}{3}\right)^2}=\frac{\frac{n}{3^{n+1}}-\frac{n+1}{3^n}+1}{\frac{4}{9}}$$

$$=\frac{9}{4}.\left\{\frac{n-3\,(n+1)+3^{n+1}}{3^{n+1}}\right\}=\frac{3^{n+1}-2n-3}{4\cdot 3^{n-1}};$$

$$\therefore \text{ sum of 10 terms}=\frac{3^{11}-20-3}{4\times 3^{10}}=\frac{3^{11}-23}{4\times 3^{10}},$$

$$\text{sum to infinity}=\frac{1}{(1-x)^2}=\frac{1}{\frac{4}{9}}=\frac{9}{4}=2\tfrac{1}{4}.$$

(f) 1. SIMPLE EQUATIONS.

The ordinary rules for the solution of simple equations will be familiar to the student, and it will not be necessary to do more than work out a few of those examples which have been given at the examinations.

Ex. 1. $$\frac{1}{x+\frac{1}{y-\frac{1}{x}}}=\frac{1}{x-\frac{1}{y-\frac{1}{x}}}. \quad . \quad (1).$$

$$\frac{1}{y}\left(1-\frac{1}{x}\right)=1. \quad . \quad . \quad . \quad (2). \qquad (1875.)$$

The first equation has $y-\frac{1}{x}$ a factor of each side;

$$\therefore y-\frac{1}{x}=0\,;\ \therefore y=\frac{1}{x}.$$

Substituting in the second equation,

$$\frac{1}{y}\,(1-y)=1\,;\ \therefore \frac{1}{y}-1=1\,;$$

$$\therefore \frac{1}{y}=2\,;\ \therefore y=\frac{1}{2},$$

and $x=\frac{1}{y}$; $\therefore$ equation (2) gives $x\left(1-\frac{1}{x}\right)=1$;

$$\therefore x=2.$$

Ex. 2. $\left(\frac{24}{x}\right)^2+(y-4)^2=65.$ . . . (1). (1875.)

$\left(\frac{12}{x}\right)^2+9=(5y-20)^2.$ . . (2).

From (2), $\left(\frac{24}{x}\right)^2-100\,(y-4)^2=-36$, by multiplying by 4.

Subtracting this from (1),

$$101\,(y-4)^2=101;$$

$$\therefore y-4=\pm 1,\ y=5 \text{ or } 3.$$

Again, multiplying equation (1) by 5^2,

$$5^2\left(\frac{24}{x}\right)^2+5^2(y-4)^2=65\times 25=1625.$$

Adding this to equation (2),

$$5^2\left(\frac{24}{x}\right)^2+\left(\frac{12}{x}\right)^2+9=65\times 25=1625;$$

$$\therefore 5^2\left(\frac{24}{x}\right)^2+\left(\frac{12}{x}\right)^2=1616;$$

$$\therefore 5^2\times 4\left(\frac{12}{x}\right)^2+\left(\frac{12}{x}\right)^2=1616;$$

$$\therefore 101\times\left(\frac{12}{x}\right)^2=1616.$$

$$\left(\frac{12}{x}\right)^2=16;\ \therefore \frac{12}{x}=\pm 4;\ \therefore x=\pm 3.$$

Ex. 3. $\frac{x}{a}+\frac{y}{b}=1-\frac{x}{c}=1+\frac{y}{c}.$ (1870.)

$$1-\frac{x}{c}=1+\frac{y}{c};\ \therefore y=-x.$$

$$\text{Again, } \frac{x}{a}+\frac{y}{b}=1-\frac{x}{c};$$

$$\therefore \frac{x}{a}-\frac{x}{b}=1-\frac{x}{c} \text{ by substitution};$$

$$\therefore x\left(\frac{1}{a}-\frac{1}{b}+\frac{1}{c}\right)=1;$$

$$\therefore x\,(bc-ac+ab)=abc;$$

$$\therefore x=\frac{abc}{bc-ac+ab}, \text{ and } y=\frac{abc}{ac-bc-ab}.$$

Ex. 4. $\frac{x}{a+2}+\frac{y}{a}=1.$ . . . . (1).

$\frac{x}{a}+\frac{y}{a-1}=1.$ . . . . (2)

From (1), $x+\dfrac{y(a+2)}{a}=a+2.$

From (2), $x+\dfrac{ay}{a-1}=a.$

Subtracting, $y\left(\dfrac{a+2}{a}-\dfrac{a}{a-1}\right)=2.$

$$y\left(\frac{a^2+a-2-a^2}{a(a-1)}\right)=2.$$

$$y=\frac{2a(a-1)}{a-2}.$$

Again, from (1), $\dfrac{ax}{a+2}+y=a.$

From (2), $\dfrac{x(a-1)}{a}+y=a-1.$

Subtracting, $x\left(\dfrac{a}{a+2}-\dfrac{a-1}{a}\right)=1,$

$$x\left(\frac{a^2-a^2-a+2}{a(a+2)}\right)=1,$$

$$x=-\frac{a(a+2)}{a-2}.$$

Ex. 5. $2x+y+z=a$. . . . . (1).
$x+2y+z=b$. . . . . (2).
$x+y+2z=c$. . . . . (3).

Adding, $4x+4y+4z=a+b+c;$

$$\therefore\ x+y+z=\frac{a+b+c}{4}.$$

Subtracting this from equation (1),

$$x=a-\frac{a+b+c}{4}=\frac{3a-b-c}{4}.$$

Subtracting it from equation (2),

$$y=b-\frac{a+b+c}{4}=\frac{3b-a-c}{4}.$$

Subtracting it from equation (3),

$$z=c-\frac{a+b+c}{4}=\frac{3c-a-b}{4}.$$

2. Quadratic Equations.

The majority of the equations given in the 1st B.A. and 1st B.Sc. examinations are quadratics. They are seldom difficult, but are usually of such a character as

to test the candidate's skill in the artifices employed in solving equations of this class.

Before studying quadratic equations, the student must thoroughly understand the method of simplifying surds, the meaning of negative and fractional indices, and the mode of extracting the square root of binomial surds. The following examples are from examination papers.

Ex. 6. Express $\frac{1-\sqrt{2}+\sqrt{5}}{1+\sqrt{2}-\sqrt{5}}$ under the form of a fraction with a rational denominator. (1872.)

$$\frac{1-\sqrt{2}+\sqrt{5}}{1+\sqrt{2}-\sqrt{5}}=\frac{(1-\sqrt{2}+\sqrt{5})(1+\sqrt{2}+\sqrt{5})}{(1+\sqrt{2}-\sqrt{5})(1+\sqrt{2}+\sqrt{5})}$$
$$=\frac{(1+\sqrt{5})^2-(\sqrt{2})^2}{(1+\sqrt{2})^2-(\sqrt{5})^2}$$
$$=\frac{1+2\sqrt{5}+5-2}{1+2\sqrt{2}+2-5}$$
$$=\frac{4+2\sqrt{5}}{2\sqrt{2}-2}=\frac{2+\sqrt{5}}{\sqrt{2}-1}$$
$$=\frac{(2+\sqrt{5})(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)}$$
$$=\frac{2\sqrt{2}+\sqrt{10}+2+\sqrt{5}}{2-1}$$
$$=2+2\sqrt{2}+\sqrt{10}+\sqrt{5}.$$

Ex. 7. Simplify

$$\frac{(a-x)^{-4}(x-a)^3}{(a-2x)^4(2x-a)^{-3}}\times\frac{(x+a)^{-3}(x-2a)^3}{(x+a)^{-4}(2a-x)^3}.$$

If a quantity in the numerator has a negative index, it may be written in the denominator with the same index written positively; and if in the denominator, it may be written in the numerator under the same conditions. The above quantity will then become

$$\frac{(x-a)^3(2x-a)^3}{(a-2x)^4(a-x)^4}\times\frac{(x-2a)^3(x+a)^4}{(2a-x)^3(x+a)^3}$$
$$=\frac{(x-a)^3(2x-a)^3}{(2x-a)^4(x-a)^4}\times-\frac{(x-2a)^3(x+a)^4}{(x-2a)^3(x+a)^3}$$
$$=\frac{1}{(2x-a)(x-a)}\times-\frac{x+a}{1}$$
$$=-\frac{x+a}{(2x-a)(x-a)}.$$

There are two methods of solving a quadratic when

it has been reduced to its simplest form—the first by completing the square, and the second by breaking the quantity up into factors when all the terms are on one side of the sign of equality, and equating each of the factors to zero.

There are two methods of completing the square—the first by making the coefficient of the highest power of the unknown quantity unity, and adding the square of one-half the coefficient of the lower power to each side of the equation; this is the usual method.

The second, called the Hindoo method, is sometimes used when the application of the preceding one would involve fractions. By this method we multiply every term in the equation by four times the coefficient of the highest power of the unknown quantity, and add the square of the coefficient of the lower to each side.

Thus, if we have $ax^2+bx=c$ for our equation, by the first method we should divide by a, and get

$$x^2+\frac{b}{a}x=\frac{c}{a}.$$

Add $\left(\frac{b}{2a}\right)^2$ to each side,

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{c}{a}+\frac{b^2}{4a^2}=\frac{4ac+b^2}{4a^2},$$

which makes the left side a complete square, of which the root is $x+\frac{b}{2a}$.

By the second method we multiply every term by $4a$.

Then $4a^2x^2+4abx=4ac$.

Adding b^2 to each side, we get

$$4a^2x^2+4abx+b^2=4ac+b^2.$$

The left side is now a complete square, of which the root is $2ax+b$.

By the first method we get

$$x+\frac{b}{2a}=\pm\frac{\sqrt{4ac+b^2}}{2a};$$

$$\therefore x=\frac{-b\pm\sqrt{4ac+b^2}}{2a}.$$

By the second method we get

$$2ax+b\pm\sqrt{4ac+b^2},$$

$$\text{or } x=\frac{-b\pm\sqrt{4ac+b^2}}{2a}, \text{ as before.}$$

The following example is worked out in each of the three ways.

Ex. 8. Find x from the equation $2x^2+9x=5$.

(1) *By factors*:—

$$2x^2+9x-5=0.$$

The coefficient of x in one of the factors must be 2, and the second terms must be 5 and 1;

$\therefore$ the only factors are $2x\pm 5$ and $x\mp 1$, or $2x\pm 1$ and $x\mp 5$.

But neither $x-1$ nor $x+1$ are factors, as the coefficients show us; and as the coefficient of x is $+9$, we should expect $x+5$ to be a factor.

Arranging, we get $2x(x+5)-(x+5)=0$;

$$\therefore (2x-1)(x+5)=0;$$

$$\therefore x+5=0, \text{ or } x=-5,$$

$$2x-1=0, \text{ or } x=+\tfrac{1}{2};$$

$$\therefore x=-5 \text{ or } \tfrac{1}{2}.$$

(2) *By completing the square in the ordinary way*:—

$$x^2+\frac{9x}{2}=\frac{5}{2};$$

$$\therefore x^2+\frac{9x}{2}+\left(\frac{9}{4}\right)^2=\frac{5}{2}+\frac{81}{16}=\frac{121}{16};$$

$$\therefore x+\frac{9}{4}=\pm\frac{11}{4};$$

$$\therefore x=\frac{2}{4}=\frac{1}{2}, \text{ or } -\frac{20}{4}=-5.$$

(3) *By the Hindoo method*:—

$$2x^2+9x=5.$$

Multiplying by 4×2, we get $16x^2+72x=40$.

Adding 9^2, $16x^2+72x+81=121$;

$$\therefore 4x+9=\pm 11;$$

$$\therefore 4x=2 \text{ or } -20,$$

$$x=\frac{1}{2} \text{ or } -5.$$

Sometimes we have equations which cannot easily be arranged so as to form a square; the method of factors must then be applied, as in the following case.

Ex. 9. Solve $\dfrac{x^2+11}{x^2+1}=\dfrac{6}{x}$.

Multiplying this up, we have $x^3+11x=6x^2+6$, or $x^3-6x^2+11x-6=0$.

This cannot be arranged as a quadratic, so the method of solution by factors *must* be applied. The only num-

bers which will divide the term independent of x are ± 1, ± 2, ± 3, ± 6; $\therefore$ the only possible factors are $x \pm 1$, $x \pm 2$, $x \pm 3$, $x \pm 6$.

If we can get *any one* of the factors out, the equation will be reduced to an ordinary quadratic.

By inspection we see that $x-1$ is a factor.

$$\text{And } x^3-6x^2+11x-6=x^2(x-1)-5x(x-1)+6(x-1)$$
$$=(x-1)(x^2-5x+6).$$
$$\therefore x-1=0, \text{ or } x=1.$$

And $x^2-5x+6=0$, which may be solved by any of the three methods.

It will be observed that the method of factors is applicable when the values of x are rational quantities only.

The following are examples from examination papers:—

Ex. 10. Solve $\dfrac{2x^2-3x+1}{x^2-2x+2}=\dfrac{2x-3}{x-2}$. (1873.)

This may be written

$$\frac{2(x^2-2x+2)+x-3}{x^2-2x+2}=\frac{2(x-2)+1}{x-2},$$
$$\text{or } 2+\frac{x-3}{x^2-2x+2}=2+\frac{1}{x-2};$$
$$\therefore \frac{x-3}{x^2-2x+2}=\frac{1}{x-2};$$
$$\therefore x^2-5x+6=x^2-2x+2;$$
$$\therefore 3x=4,\ x=1\tfrac{1}{3}.$$

Ex. 11. Solve $\dfrac{1}{\sqrt{x}-\sqrt{2-x}}-\dfrac{1}{\sqrt{x}+\sqrt{2-x}}=1$.

$$\text{Here } \frac{\sqrt{x}+\sqrt{2-x}}{x-(2-x)}-\frac{\sqrt{x}-\sqrt{2-x}}{x-(2-x)}=1;$$
$$\therefore \frac{\sqrt{x}+\sqrt{2-x}-\sqrt{x}+\sqrt{2-x}}{2x-2}=1;$$
$$\therefore \sqrt{2-x}=2x-2;$$
$$\therefore 2-x=4x^2-8x+4;$$
$$\therefore 4x^2-7x+2=0;$$
$$\therefore x^2-\frac{7x}{4}=-\frac{1}{2},$$
$$x^2-\frac{7x}{4}+\left(\frac{7}{8}\right)^2=\left(\frac{7}{8}\right)^2-\frac{1}{2}$$
$$=\frac{49}{64}-\frac{32}{64}=\frac{17}{64};$$

$$\therefore\ x-\frac{7}{8}=\pm\frac{1}{8}\sqrt{17};$$

$$x=\frac{1}{8}(7\pm\sqrt{17}).$$

Ex. 12. Solve $\dfrac{x^2+x+\frac{1}{2}}{a^2+1}+\dfrac{x^2+x}{a^2-1}=0.$

This may be written, $\dfrac{x^2+x+\frac{1}{2}}{x^2+x}=-\left(\dfrac{a^2+1}{a^2-1}\right).$

Taking the sum of numerator and denominator for a new numerator, and the difference for a new denominator, we get

$$\frac{2x^2+2x+\frac{1}{2}}{\frac{1}{2}}=-\left\{\frac{2a^2}{2}\right\}=-a^2;$$

$$\therefore\ 4x^2+4x+1=-a^2.$$

Extracting the root, $2x+1=\pm a\sqrt{-1};$

$$\therefore\ 2x=-1\pm a\sqrt{-1};$$

$$\therefore\ x=-\frac{1}{2}(1\mp a\sqrt{-1}).$$

Ex. 13. Find x from the equation,

$$\frac{x+\frac{1}{x}-1}{x-\frac{1}{x}+1}=1-\left(x-\frac{1}{x}\right).\quad (1868.)$$

This becomes $\dfrac{x^2-x+1}{x^2+x-1}=1-\left(\dfrac{x^2-1}{x}\right),$

$$\text{or } \frac{(x^2+x-1)-2x+2}{x^2+x-1}=1-\frac{x^2-1}{x},$$

$$\text{or } 1-\frac{2x-2}{x^2+x-1}=1-\frac{x^2-1}{x};$$

$$\therefore\ \frac{2(x-1)}{x^2+x-1}=\frac{x^2-1}{x};$$

$$\therefore\ \frac{2}{x^2+x-1}=\frac{x+1}{x};$$

$$\therefore\ x-1=0,\text{ or } x=1,$$

$$\begin{aligned}\text{and } 2x&=(x+1)(x^2+x-1)\\ &=(x^3+x^2-x+x^2+x-1)\\ &=x^3-2x^2+1;\end{aligned}$$

$$\therefore x^3+2x^2-2x-1=0;$$
$$\therefore x^2(x-1)+3x(x-1)+(x-1)=0;$$
$$\therefore x-1=0,\ x=1, \text{ and } x^2+3x+1=0;$$
$$\therefore x^2+3x+\left(\frac{3}{2}\right)^2=\left(\frac{3}{2}\right)^2-1=\frac{9-4}{4}=\frac{5}{4};$$
$$\therefore x+\frac{3}{2}=\frac{1}{2}\sqrt{5},$$
$$x=\frac{1}{2}(\pm\sqrt{5}-3).$$

The following are examples which are reduced to quadratics by arranging the terms in a particular manner:—

Ex. 14. $\sqrt{x^2-8x+31}+(x-4)^2=5;$ (1874.)

$$\therefore \sqrt{x^2-8x+31}+x^2-8x+16=5;$$
$$\therefore \sqrt{x^2-8x+31}+(x^2-8x+31)=20.$$

If we substitute y for $\sqrt{x^2-8x+31}$, we get

$$y+y^2=20,$$
$$y^2+y+\frac{1}{4}=\frac{81}{4},$$
$$y+\frac{1}{2}=\pm\frac{9}{2};\ \therefore y=\frac{8}{2}=4, \text{ or } -\frac{10}{2}=-5;$$
$$\therefore \sqrt{x^2-8x+31}=4, \text{ or } -5;$$
$$\therefore x^2-8x+31=16, \text{ or } 25;$$
$$\therefore x^2-8x=-15, \text{ or } -6;$$
$$\therefore x^2-8x+15=0, \text{ or } (x-3)(x-5)=0;$$
$$\therefore x=3 \text{ or } 5, \text{ and } x^2-8x=-6,$$
$$x^2-8x+16=10,$$
$$x-4=\pm\sqrt{10};\ \therefore x=4\pm\sqrt{10}.$$

Ex. 15. Solve $x^2+\sqrt{x}=\frac{72}{x}$.

From this, $x^3+x\sqrt{x}=72$.

Put $y=x\sqrt{x}$;

$$\therefore y^2+y=72,$$
$$y^2+y+\frac{1}{4}=\frac{289}{4},$$
$$y+\frac{1}{2}=\pm\frac{17}{2};$$
$$\therefore y=\frac{16}{2} \text{ or } -\frac{18}{2}=8 \text{ or } -9;$$
$$\therefore x\sqrt{x}=8, \text{ or } -9;$$
$$\therefore x=4, \text{ or } -3\sqrt[3]{3}.$$

Ex. 16. Solve $\{(x+3)^2+x+3\}^2-7(x+3)^2$
$=711-7x.$

This may be arranged

$$\{(x+3)^2+(x+3)\}^2-7\{(x+3)^2+(x+3)\}=690.$$

Let $y=\{(x+3)^2+(x+3)\}$;

$$\therefore y^2-7y=690;$$

$$\therefore y^2-7y+\left(\frac{7}{2}\right)^2=690+\frac{49}{4}=\frac{2809}{4};$$

$$\therefore y-\frac{7}{2}=\pm\frac{53}{2};$$

$$\therefore y=30, \text{ or } -23;$$

$$\therefore \{(x+3)^2+(x+3)\}=30, \text{ or } -23;$$

$$\therefore (x+3)^2+(x+3)=30, \text{ or } -23.$$

Let $y=x+3$;

$$\therefore y^2+y=30, \text{ or } -23;$$

$$\therefore y^2+y+\frac{1}{4}=\frac{121}{4}, \text{ or } -\frac{91}{4};$$

$$\therefore y+\frac{1}{2}=\pm\frac{11}{2}, \text{ or } \pm\frac{1}{2}\sqrt{-91};$$

$$\therefore y=5, \text{ or } -6, \text{ or } \pm\frac{1}{2}\sqrt{-91};$$

$$\therefore x+3=5, \text{ or } -6, \text{ or } \pm\frac{1}{2}\sqrt{-91};$$

$$\therefore x=2, \text{ or } -9, \text{ or } -3\pm\frac{1}{2}\sqrt{-91}.$$

Ex. 17. Solve $\dfrac{x}{x+4}+\dfrac{4}{\sqrt{x+4}}=\dfrac{21}{x}.$

This may be written $\dfrac{x^2}{x+4}+\dfrac{4x}{\sqrt{x+4}}=21$;

$$\therefore \frac{x^2}{x+4}+\frac{4x}{\sqrt{x+4}}+4=25;$$

$$\therefore \frac{x}{\sqrt{x+4}}+2=\pm 5;$$

$$\therefore \frac{x}{\sqrt{x+4}}=3, \text{ or } -7;$$

$$\therefore \frac{x^2}{x+4}=9, \text{ or } 49,$$

$$x^2=9x+36, \text{ or } 49x+196,$$

$$x^2-9x-36=0; \therefore (x+3)(x-12)=0;$$

$$\therefore x=-3, \text{ or } +12,$$

$$x^2-49x=196,$$

$$x^2-49x+\left(\frac{49}{2}\right)^2=\left(\frac{49}{2}\right)^2+196=\frac{3185}{4};$$

$$\therefore x-\frac{49}{2}=\pm\frac{1}{2}\sqrt{3185}\,;\ \therefore x=\frac{1}{2}\,(49\pm\sqrt{3185}).$$

The following artifice is frequently useful in arranging an equation which has more than two powers of x.

Ex. 18. Solve $x^4-8x^3+10x^2+24x+5=0$.

It will be observed that the first two terms are the same as those of $(x-2)^4$.

Expanding $(x-2)^4$, we get

$$x^4-8x^3+24x^2-32x+16;$$

that is,

$$(x^2-8x^3+10x^2+24x+5)+14x^2-56x+11\,;$$

$$\therefore (x-2)^4=(x^2-8x^3+10x^2+24x+5)+14x^2-56x+11.$$

The part on the right in brackets is the given equation, and is equal to zero by the question;

$$\begin{aligned}\therefore (x-2)^4&=0+14x^2-56x+11\\&=14x\,(x-4)+11.\end{aligned}$$

Let $y=x-2$. Then $x=y+2$, and $x-4=y-2$;

$$\begin{aligned}\therefore \text{by substitution, } y^4&=14\,(y+2)\,(y-2)+11\\&=14\,(y^2-4)+11\,;\\ \therefore y^4&=14y^2-56+11\\&=14y^2-45\,;\end{aligned}$$

$$\therefore y^4-14y+45=0\,;\ \therefore (y^2-9)\,(y^2-5)=0\,;$$

$$\therefore y=\pm 3,\text{ or }\pm\sqrt{5}.$$

Now, putting $x-2$ for y, we get

$$\left.\begin{aligned}x-2&=\pm 3\,;\ \therefore x=5\text{ or }-1,\\ x-2&=\pm\sqrt{5}\,;\ \therefore x=2\pm\sqrt{5}\end{aligned}\right\}.$$

Ex. 19. Solve $(x-a)^2+2\sqrt{x}\,(x-a)=a^2+\sqrt{x}$.

This may be written

$$(x-a)^2+2\sqrt{x}\left(x-a-\frac{1}{2}\right)=a^2\,;$$

that is,

$$\left(x-a-\frac{1}{2}\right)^2+2\sqrt{x}\left(x-a-\frac{1}{2}\right)-\frac{1}{4}+x-a=a^2,$$

the quantities after the second bracket being added to cancel the difference between $(x-a)^2$ and $\left(x-a-\frac{1}{2}\right)^2$;

$$\therefore \left(x-a-\frac{1}{2}\right)^2+2\sqrt{x}\left(x-a-\frac{1}{2}\right)+x=a^2+a+\frac{1}{4}.$$

This is a quadratic, for if y be put for $x-a-\frac{1}{2}$, the equation would become $(y+\sqrt{x})^2=a^2+a+\frac{1}{4}$;

$$\therefore \left(x-a-\frac{1}{2}\right)+\sqrt{x}=\sqrt{a^2+a+\frac{1}{4}}=a+\frac{1}{2};$$

$$\therefore x-a+\sqrt{x}=1;$$

$$\therefore x+\sqrt{x}=a+1,$$

$$x+\sqrt{x}+\frac{1}{4}=2a+1\frac{1}{4};$$

$$\therefore \sqrt{x}+\frac{1}{2}=\pm\sqrt{2a+1\frac{1}{4}};$$

$$\therefore \sqrt{x}=\pm\sqrt{2a+1\frac{1}{4}}-\frac{1}{2};$$

$$\therefore x=\left\{\sqrt{2a\pm 1\frac{1}{4}}-\frac{1}{2}\right\}^2$$

$$=2a+1\frac{1}{4}+\frac{1}{4}\pm\sqrt{2a\pm 1\frac{1}{4}}$$

$$=2a+1\frac{1}{2}\pm\sqrt{2a\pm 1\frac{1}{4}}.$$

Ex. 20. Solve $x^2+x^{-2}+x+x^{-1}=4$.

$$x^2+x^{-2}=(x+x^{-1})^2-2;$$

$\therefore$ the equation becomes

$$(x+x^{-1})^2-2+x+x^{-1}=4;$$

$$\therefore (x+x^{-1})^2+(x+x^{-1})=6,$$

$$(x+x^{-1})^2+(x+x^{-1})+\frac{1}{4}=\frac{25}{4};$$

$$\therefore (x+x^{-1})+\frac{1}{2}=\pm\frac{5}{2},$$

$$(x+x^{-1})=\frac{4}{2}, \text{ or } 2=-\frac{6}{2}, \text{ or } -3;$$

$$\therefore \frac{x^2+1}{x}=2, \text{ or } -3;$$

$$\therefore x^2+1=2x, \text{ or } -3x;$$

$$\therefore x^2-2x+1=0;\ \therefore x-1=0;\ \therefore x=1$$

and $x^2+3x=-1$;

$$\therefore x^2+3x+\left(\frac{3}{2}\right)^2=\frac{9}{4}-1=\frac{5}{4};$$

$$\therefore x+\frac{3}{2}=\pm\frac{1}{2}\sqrt{5};$$

$$\therefore x=-\frac{1}{2}\pm(3\sqrt{5}).$$

Instead of arranging after the method of the above examples, a factor may sometimes be taken out if a particular method of arrangement is adopted.

Ex. 21. Solve $x^2-\frac{2}{3x}=1\frac{4}{9}$.

This may be written $x^2-\frac{4}{9}=\frac{2}{3x}+1$

$$=\frac{1}{x}\left(x+\frac{2}{3}\right).$$

$x+\frac{2}{3}$ will now divide both sides of the equation;

$$\therefore x+\frac{2}{3}=0, \text{ and } x=-\frac{2}{3}.$$

$$\text{Also } x-\frac{2}{3}=\frac{1}{x};$$

$$\therefore x^2-\frac{2x}{3}=1;$$

$$\therefore x^2-\frac{2x}{3}+\left(\frac{1}{3}\right)^2=1\frac{1}{9}=\frac{10}{9};$$

$$\therefore x-\frac{1}{3}=\pm\frac{1}{3}\sqrt{10};$$

$$\therefore x=\frac{1}{3}(1\pm\sqrt{10}).$$

Ex. 22. Solve $x^n+4x^{-n}=a^n$.

$$\text{Here we have } \frac{x^{2n}+4}{x^n}=a^n,$$

$$\text{or } x^{2n}-a^nx^n+4=0;$$

$$\therefore x^{2n}-a^nx^n=4;$$

$$\therefore x^{2n}-a^nx^n+\left(\frac{a^n}{2}\right)^2=\left(\frac{a^n}{2}\right)^2+4$$

$$=\frac{a^{2n}+16}{4};$$

$$\therefore x^n-\frac{a^n}{2}=\pm\frac{1}{2}\sqrt{a^{2n}+16};$$

$$\therefore x^n = \frac{1}{2}\{a^n \pm \sqrt{a^{2n}+16}\};$$

$$\therefore x = \sqrt[n]{\frac{1}{2}\{a^n \pm \sqrt{a^{2n}+16}\}}.$$

The following examples are worked out as quadratics.

Ex. 23. $9^x + 3^{x+1} = 108.$

$$9^x = 3^{2x}, \text{ and } 3^{x+1} = 3 \times 3^x;$$

$\therefore$ the equation becomes $3^{2x} + 3 \cdot 3^x = 108.$

Computing the square,

$$3^{2x} + 3 \cdot 3^x + \left(\frac{3}{2}\right)^2 = 108 + \frac{9}{4}$$

$$= \frac{441}{4};$$

$$\therefore 3^x + \frac{3}{2} = \frac{21}{2};$$

$$\therefore 3^x = \frac{18}{2} = 9$$

$$= 3^2;$$

$$\therefore x = 2.$$

It will be worth the student's trouble to master the chapter on 'The Theory of Quadratic Equations' in Todhunter's *Algebra for Colleges and Schools.* He should be familiar with the values of x in the general quadratic equation $ax^2 + bx + c = 0$, viz. $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$ It is also important to remember that if this equation be reduced to $x^2 + \dfrac{bx}{a} + \dfrac{c}{a} = 0$, the sum of the values of x which satisfy it $= -\dfrac{b}{a}$, and their product $\dfrac{c}{a}$. These facts will be self-evident on considering the mode of solving the equation by finding its factors.

If one value of x is given, the other may be found from the facts above mentioned; for let $x = m$ be one value of x, then the other value will be $-\dfrac{b}{a} - m$, or $\dfrac{c}{am}$.

We may also write an equation which shall have given values of x; for if $x = m$ and $x = n$ are the given values, the coefficient of x, with its sign changed, namely, $-\dfrac{b}{a} = m + n$, and the independent term $\dfrac{c}{a} = mn$, from

these values m and n may be found and substituted in $ax^2+bx+c=0$.

Or we may proceed thus: If $x=m$ and $x=n$ be the values of x, $x-m=0$, and $x-n=0$;

$$\therefore (x-m)(x-n)=0;$$

$\therefore x^2-(m+n)x+mn=0$ will be the required equation.

If $x=m$ and $x=n$ are the values of x in the equation $ax^2+bx+c=0$, we may deduce from it an equation having $\frac{1}{m}, \frac{1}{n}$ for the values of x.

Let $x=\frac{1}{m}$; $\therefore m=\frac{1}{x}$, and $x=\frac{1}{n}$; $\therefore n=\frac{1}{x}$.

If now we substitute $\frac{1}{x}$ for x in the given equation, we shall have a new equation, in which the values of x will be the reciprocals of their former values—that is,

$a.\frac{1}{x^2}+\frac{b}{x}+c=0$;

$$\therefore a+bx+cx^2=0 \text{ is the equation.}$$

Ex. 24. Find the sum, difference, and product of the values of x in the equation $x^2-42x+117=0$.

Here $\frac{b}{a}=-42$; $\therefore -\frac{b}{a}=42$, the sum,

and $\frac{c}{a}=117$; $\therefore 117$ is the product.

The difference may be found from the values of x in the general equation

$$x=-\frac{b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{(-b+\sqrt{b^2-4ac})-(-b^2-\sqrt{b^2-4ac})}{2a}$$

$$=\frac{\sqrt{b^2-4ac}}{a}.$$

Now $-\frac{b}{a}=42$, $\frac{c}{a}=117$; $\therefore a=1$,

$b=-42$,

$c=117$;

$$\therefore \frac{\sqrt{b^2-4ac}}{a}=\sqrt{1296}=36, \text{ the difference;}$$

$\therefore$ the values of x are -3 and -39.

Ex. 25. Find the condition that the quadratic equation $ax^2+2bx+c=0$ may have equal roots. (1875.)

Let p, p be the equal roots.

Then the equation having p, p for the values of x, will be $(x-p)(x-p)=0$, or $x^2-2p+p^2=0$;

$$\therefore\ -2p=2\frac{b}{a};\ \therefore p=-\frac{b}{a},\ \text{and } p^2=\frac{c}{a};$$

$$\therefore \frac{b^2}{a^2}=\frac{c}{a};\ \therefore b^2=ac, \text{ the condition required.}$$

Ex. 26. Write down a quadratic equation which has its roots $\alpha+\beta$ and $\alpha-\beta$. (1875.)

If these are the values of x,

$$\{x-(\alpha+\beta)\}\{x-(\alpha-\beta)\}=0,$$
$$\text{or } \{(x-\alpha)-\beta\}\{(x-\alpha)+\beta\}=0;$$
$$\therefore (x-\alpha)^2-\beta^2=0;$$
$$\therefore x^2-2\alpha x+\alpha^2-\beta^2=0 \text{ is the required equation.}$$

When equations like $x^2=1, x^2=-1, x^3=\pm 1, x^4=\pm 1$, are given, care must be taken to find all the values of x, and it must be remembered that unity has two square roots, three cube roots, four fourth roots, all different, and so n, nth roots, all different, as the following examples show:—

$$\text{Let } x^2=1;\ \therefore x^2-1=0;\ \therefore (x+1)(x-1)=0;$$
$$\therefore x=\pm 1.$$
$$\text{Let } x^2=-1;\ \therefore x=\pm\sqrt{-1}.$$
$$\text{Let } x^3=1;\ \therefore x^3-1=0;\ \therefore (x-1)(x^2+x+1)=0;$$
$$\therefore x=1, \text{ and } x^2+x+1=0;$$
$$\therefore x^2+x+\frac{1}{4}=-\frac{3}{4};$$
$$\therefore x+\frac{1}{2}=\pm\frac{1}{2}\sqrt{-3},\ x=\frac{1}{2}(-1\pm\sqrt{-3}).$$
$$\text{Let } x^4=1;\ \therefore x^4-1=0;\ \therefore (x^2-1)(x^2+1)=0;$$
$$\therefore (x+1)(x-1)(x^2+1)=0;$$
$$\therefore x=\pm 1, \text{ and } x^2+1=0;$$
$$\therefore x=\pm\sqrt{-1}.$$

If $x^2=1$, the values of x are $x=\pm 1$, which are the two square roots of unity.

If $x^3=1$, $x=1$, or $\frac{1}{2}(-1\pm\sqrt{-3})$, which are the three cube roots of unity.

If $x^4=1$, $x=\pm 1$, and $\pm\sqrt{-1}$, which are the four fourth roots of unity.

Quadratic Equations of more than One Unknown Quantity.

Simultaneous quadratic equations occur in such a variety of forms, that it is impossible to give rules for their simplification which will apply to more than a few of them at most.

The forms below are the chief objects of search; and when we obtain them, the solution is simple enough, viz. the values of—

1. $x+y$ and $x-y$.	4. $x-y$ and xy.
2. x^2+y^2 and x^2-y^2.	5. x^2+y^2 and xy.
3. $x+y$ and xy.	6. x^2-y^2 and $x\pm y$.

The chief difficulty in the solution of quadratics is in getting these forms. Before giving a few general hints on the method of reducing given equations to these forms, we will show how the values of x and y may be obtained when any pair of these forms is arrived at.

1. If we have $x+y$ and $x-y$, we add them together, and divide by 2 for x. Subtract them and divide by 2 for y.

2. If x^2+y^2 and x^2-y^2 are obtained, we proceed in the same manner, but extract the square root in each case after dividing by 2.

3. Let $x+y=a$. . . . . . . (1.)

$$xy=b. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

Square (1.); $\therefore x^2+2xy+y^2=a^2$.

Multiply (2.) by -4; $-4xy=-4b$.

Adding, $x^2-2xy+y^2=a^2-4b^2$;

$$\therefore x-y=\sqrt{a^2-4b^2}.$$

This will now be solved as case 1.

4. If we have $x-y$ and xy, the method will be the same, except that we multiply the second equation by $+4$.

5. If we have x^2+y^2 and xy, we proceed thus:

Let $x^2+y^2=a$. . . . (1.)

$$xy=b. \quad . \quad . \quad . \quad (2.)$$

Multiply (2.) by ± 2;

$$\therefore \pm 2xy = \pm 2b.$$

Taking the upper sign and adding, we get

$$x^2+2xy+y^2=a+2b; \therefore x+y=\pm\sqrt{a+2b}.$$

Taking the lower sign and adding,

$$x^2-2xy+y^2=a-2b; \therefore x-y=\pm\sqrt{a-2b}.$$

6. Let $x^2-y^2=a$. . . . . . . (1.)

$x+y=b$. . . . . . . (2.)

Divide (1.) by (2.); $\therefore \dfrac{x^2-y^2}{x+y}=\dfrac{a}{b}$; $\therefore x-y=\dfrac{a}{b}$.

This result and equation (2.) reduce the equations to the first case.

If x^2-y^2 and $x-y$ are obtained, the process is precisely similar.

We now proceed to show by means of examples the usual artifices employed in reducing simultaneous quadratic equations to one of these forms.

Ex. 1. Solve $x+y=\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{5}{2}$.

$$\text{From this } x+y=\frac{5}{2}, \text{ and } \frac{x+y}{xy}=\frac{5}{2}.$$

It is therefore evident that $xy=1$.

This gives us the pair of equations corresponding in form with those of case 3, from which

$$x=2 \text{ or } \frac{1}{2}, \text{ and } y=\frac{1}{2} \text{ or } 2.$$

Ex. 2. Solve $\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{x+y}{12}=\dfrac{7}{x+y+5}$.

Since $\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{x+y}{12}$; $\therefore \dfrac{x+y}{xy}=\dfrac{x+y}{12}$; $\therefore xy=12$.

$$\text{And } \frac{1}{x}+\frac{1}{y}=\frac{7}{x+y+5};$$

$$\therefore \frac{x+y}{xy}=\frac{7}{x+y+5}.$$

Substituting 12 for xy, $\dfrac{x+y}{12}=\dfrac{7}{x+y+5}$.

Multiplying up, $(x+y)^2+5(x+y)=84$, which is a quadratic in $(x+y)$.

Completing the square,

$$(x+y)^2+5(x+y)+\left(\frac{5}{2}\right)^2=84+\frac{25}{4}=\frac{361}{4};$$

$$\therefore (x+y)+\frac{5}{2}=\pm\frac{19}{2};$$

$$\therefore x+y=7 \text{ or } -12.$$

And since $xy=12$, we have another example of case 3.

Ex. 3. Solve $\frac{x^2}{y}-\frac{y^2}{x}=28$, ... (1.)

and $x-y=8$. . . . . . (2.)

From (1.) $\frac{x^3-y^3}{xy}=28$.

Dividing this result by equation (2.),

$$\frac{x^2+xy+y^2}{xy}=\frac{28}{8}=\frac{7}{2}.$$

Adding numerator and denominator together for a new numerator,

$$\frac{x^2+2xy+y^2}{xy}=\frac{9}{2};$$

$$\therefore \frac{x^2+2xy+y^2}{-4xy}=\frac{9}{-8}.$$

Adding numerator and denominator for a new denominator,

$$\frac{x^2+2xy+y^2}{x^2-2xy+y^2}=\frac{9}{1};$$

$$\therefore \frac{x+y}{x-y}=\pm 3;$$

$$\therefore x+y=3x-3y, \text{ or } 3y-3x;$$

$$\therefore 4y=2x, \text{ or } 2y=4x;$$

$$\therefore x=2y, \text{ or } \frac{y}{2}.$$

Substituting in equation (2.), we get

$$y=8, \text{ or } -16,$$
$$x=16, \text{ or} -8.$$

Ex. 4. Solve $x^2+y^2+x+y=330$. . . (1.)

$x^2-y^2+x-y=150$. . . (2.)

Adding these together, we get

$$2x^2+2x=480;$$

$$\therefore x^2+x=240,$$

a quadratic of one unknown quantity only.

Ex. 5. Solve $x^4=3x+2y$. . . . . . (1.)

$y^4=2x-3y$. . . . (2.)

Subtracting, $x^4-y^4=x-y$.

Dividing out the common factor $x-y$, we get

$$x-y=0; \therefore x=y, \text{ and } (x^2+y^2)(x+y)=1;$$

$$\therefore x^3+x^2y+xy^2+y^3=1.$$

Substituting $x=y$ in equation (1.),

$$x^4=3x+2x=5x;$$

$$\therefore x=\sqrt[3]{5}, y=\sqrt[3]{5}.$$

This gives only one value of x and y, while the original equations show us there should be four values of each. The remaining values will be found from

$$x^3+x^2y+xy^2+y^3=1,$$

and the original equations.

Ex. 6. Solve $x+y=3$. . . . . . (1.)

$$x^5+y^5=33. \quad . \quad . \quad . \quad . \qquad (2.)$$

Dividing (2.) by (1.),

$$x^4-x^3y+x^2y^2-xy^3+y^4=11.$$

By (1.), $(x+y)^4=x^4+4x^3y+6x^2y^2+4xy^3+y^4=81.$

$$\text{Subtracting, } 5x^3y+5x^2y^2+5xy^3=70;$$
$$\therefore xy(x^2+xy+y^2)=14;$$
$$\therefore xy\{(x+y)^2-xy\}=14.$$
$$\text{But } x+y=3; \therefore xy(3^2-xy)=14;$$
$$\therefore 9xy-x^2y^2=14;$$
$$\therefore x^2y^2-9xy+14=0;$$
$$\therefore (xy-2)(xy-7)=0;$$
$$\therefore xy=2 \text{ or } 7.$$

This result, with equation (1.), is identical with case 3.

If we have xy and the sum or difference of two high powers given, the method of substitution is sometimes most expeditious.

Ex. 7. Solve $xy=12$. . . . . . (1.)

$$x^5-y^5=781. \quad . \quad . \quad . \qquad (2.)$$

$$\text{From (1.), } x=\frac{12}{y}.$$

$$\text{Substituting in (2.), } \left(\frac{12}{y}\right)^5-y^5=781;$$
$$\therefore 12^5-y^{10}=781y^5;$$
$$\therefore y^{10}+781y^5=12^5,$$

which may be solved like an ordinary quadratic of one unknown quantity.

Ex. 8. Solve $xy(x^2+y^2)=3$. . . . . (1.)

$$x^2y^2(x^4+y^4)=7. \quad . \quad . \quad . \qquad (2.)$$

$$\text{Squaring (1.), } x^2y^2(x^4+y^4+2x^2y^2)=9;$$
$$\therefore x^2y^2(x^4+y^4)+2x^4y^4=9.$$

Substituting the value of $x^2y^2(x^4+y^4)$ from (2.), $7+2x^4y^4=9$;

$$\therefore x^4y^4=1; \therefore xy=1.$$

And from (1.) by substitution, $x^2+y^2=3$, which is like case 5.

Ex. 9. Solve $x^2+xy+y^2=5$. . . . . (1.)

$$x^4+x^2y^2+y^4=11. \quad . \quad . \quad . \qquad (2.)$$

Dividing (2.) by (1.), $x^2-xy+y^2=\frac{11}{5}$.

By (1.), $x^2+xy+y^2=5$;

$$\therefore\ x^2+y^2=\frac{1}{2}\left(\frac{11}{5}+5\right)=\frac{18}{5}.$$

Subtracting, $2xy=\frac{14}{5}$, which is a similar case to the above.

Ex. 10. Solve $y^4-432=12xy^2$. . . . (1.)

$y^2-12=2xy$. . . . (2.)

Multiplying (2.) by $6y$, $6y^3-72y=12xy^2$.

Equating this with left side of equation (1.),

$$y^4-432=6y^3-72y;$$
$$\therefore\ y^4-6y^3+72y-432=0;$$
$$\therefore\ y^3(y-6)+72(y-6)=0;$$
$$\therefore\ y=6,$$
$$y^3+72=0;$$
$$\therefore\ y=\sqrt[3]{-72}=-2\sqrt[3]{9}.$$

And the values of x may be found by substitution.

Ex. 11. Solve $x^3+y^3=a$. . . . . (1.)

$xy(x+y)=b$. . . . (2.)

Multiplying (2.) by 3;

$$\therefore\ 3x^2y+3xy^2=3b.$$

Adding this to (1.),

$$x^3+3x^2y+3xy^2+y^3=a+3b;$$
$$\therefore\ x+y=\sqrt[3]{a+3b}.$$

Substituting in (2.),

$$xy(\sqrt[3]{a+3b})=b;$$
$$\therefore\ xy=\frac{b}{\sqrt[3]{a+3b}},$$

which is like case 3.

Ex. 12. Solve $x+\sqrt{xy}=a$. . . . (1.)

$y+\sqrt{xy}=b$. . . . (2.)

Adding these together, $x+2\sqrt{xy}+y=a+b$.

Taking square root, $\sqrt{x}+\sqrt{y}=\sqrt{a+b}$.

By (1.), $\sqrt{x}(\sqrt{x}+\sqrt{y})=a$;

$$\therefore\ \sqrt{x}(\sqrt{a+b})=a;\ \therefore\ \sqrt{x}=\frac{a}{\sqrt{a+b}};$$
$$\therefore\ x=\frac{a^2}{a+b}.$$

By (2.), $\sqrt{y}(\sqrt{x}+\sqrt{y})=b$; $\therefore\ \sqrt{y}=\dfrac{b}{\sqrt{a+b}}$;

$$\therefore\ y=\frac{b^2}{a+b}.$$

Ex. 13. Solve $x+y=80$. . . . . (1.)

$$\sqrt{xy}-\sqrt{x}-\sqrt{y}=20. \quad . \quad . \quad (2.)$$

Multiplying (2.) by 2,

$$2\sqrt{xy}-2(\sqrt{x}+\sqrt{y})=40.$$

Adding this to (1.),

$$(x+2\sqrt{xy}+y)-2(\sqrt{x}+\sqrt{y})=120.$$

Adding unity,

$$(\sqrt{x}+\sqrt{y})^2-2(\sqrt{x}+\sqrt{y})+1=121;$$
$$\therefore\ (\sqrt{x}+\sqrt{y})-1=\pm 11;$$
$$\therefore\ \sqrt{x}+\sqrt{y}=12 \text{ or } -10;$$
$$\therefore\ x+2\sqrt{xy}+y=144 \text{ or } 100.$$

Substituting from (1.),

$$80+2\sqrt{xy}=144 \text{ or } 100;$$
$$\therefore\ 2\sqrt{xy}=64 \text{ or } 20.$$

Subtracting this result from (1.),

$$(\sqrt{x}-\sqrt{y})^2=16 \text{ or } 60;$$
$$\therefore\ \sqrt{x}-\sqrt{y}=\pm 4 \text{ or } \pm 2\sqrt{15}.$$
$$\text{And } \sqrt{x}+\sqrt{y}=12 \text{ or } -10;$$
$$\therefore\ 2\sqrt{x}=14 \text{ or } 10 \text{ or } -10\pm 2\sqrt{15};$$
$$\therefore\ \sqrt{x}=8 \text{ or } 4 \text{ or } -5\pm\sqrt{15};$$
$$\therefore\ x=64 \text{ or } 16 \text{ or } (-5\pm\sqrt{15})^2;$$
$$\therefore\ y=16 \text{ or } 64.$$

Ex. 14. Find x and y, having given

$$x^{x+y}=y^4. \quad . \quad . \quad . \quad (1.)$$
$$y^{x+y}=x. \quad . \quad . \quad . \quad (2.)$$

$$\text{From (1.) } x=y^{\frac{4}{x+y}}.$$
$$\text{,, (2.) } x=y^{x+y};$$
$$\therefore\ y^{\frac{4}{x+y}}=y^{x+y}.$$

The exponents must therefore be equal;

$$\therefore\ \frac{4}{x+y}=x+y;$$
$$\therefore\ x+y=\pm 2.$$

By (1.) and substitution, $x^{\pm 2}=y^4$;

$$\therefore\ x^{\pm 1}=y^2;$$

$$\therefore\ x=y^2 \text{ or } \frac{1}{x}=y^2.$$

$$\text{But } x+y=\pm 2;$$

$\therefore\ y^2+y=\pm 2$, from which y and then x may be found.

It is sometimes advisable to substitute some value of one unknown quantity for the other, and then find the quantity which expresses this relation. For example:

Ex. 15. Solve $x^2+xy+4y^2=6$ . . . . (1.)
$3x^2+8y^2=14$ . . . . (2.)

Let $y=vx$.

Equation (1.) then becomes

$$x^2+vx^2+4v^2x^2=6,$$

and equation (2.) becomes

$$3x^2+8v^2x^2=14;$$

$$\therefore\ \frac{x^2+vx^2+4v^2x^2}{3x^2+8v^2x^2}=\frac{6}{14}=\frac{3}{7};$$

$$\therefore\ \frac{1+v+4v^2}{3+8v^2}=\frac{3}{7};$$

$$\therefore\ 7+7v+28v^2=9+24v^2;$$

$$\therefore\ 7v+4v^2=2;$$

$$\therefore\ 4v^2+7v=2,$$

$$\text{or } 4v^2+7v-2=0;$$

$$\therefore\ (4v-1)(v+2)=0;$$

$$\therefore\ v=-2 \text{ or } \frac{1}{4};$$

$$\therefore\ y=-2x \text{ or } \frac{x}{4},$$

which being substituted in one of the given equations will make the solution very simple.

Many other substitutions are used to reduce the labour of solution. The following is another instance:—

Ex. 16. Solve $x+y=3$. . . . . (1.)
$x^5+y^5=33$. . . . . (2.)

Let $v=x-y$;

$$\therefore\ x=\frac{3+v}{2},\ y=\frac{3-v}{2}.$$

Substituting in (2.),

$$\frac{(3+v)^5+(3-v)^5}{2^5}=33, \text{ or}$$

$$\frac{\begin{array}{c}(3^5+5\cdot 3^4v+10\,.\,3^3v^2+10\,.\,3^4v^3+5\cdot 3\cdot v^4+v^5)\\ +(3^5-5\cdot 3^4v+10\,.\,3^3v^2-10\,.\,3^2v^3+5\cdot 3v^4-v^5)\end{array}}{2^5}$$

$$=33;$$

$$\therefore\ 2\cdot 3^5+20\,.\,3^3v^2+10\,.\,3\cdot v^4=2^5\times 33;$$

$$\therefore\ 3^4+10\times 3^2v^2+5\,.\,v^4=2^4\times 11;$$

$$\therefore\ 81+90v^2+5v^4=176;$$

$$\therefore\ 90v^2+5v^4=95;$$

$$\therefore\ v^4+18v^2=19.$$

This is a quadratic in v^2, and its factors are

$$(v^2+19)\,(v^2-1);$$

$$\therefore\ v^2=-19, \text{ or } v^2=1;$$

$$\therefore\ v=\pm 1;$$

$$\therefore\ x=y\mp 1=3-y \text{ by (1.)};$$

$$\cdot\ 2y=3\pm 1=2 \text{ or } 4;$$

$$\therefore\ y=1 \text{ or } 2;$$

$$\therefore\ x=2 \text{ or } 1,$$

$$y=3.$$

Ex. 17. Solve

$$x+\sqrt{3y^2-11+2x}=7+2y-y^2. \qquad (1.)$$

$$y^2=\frac{2(x+y)-(x-y)^2}{x-y}. \qquad (2.)$$

Multiply the first by 2;

$$\therefore\ 2x+2\sqrt{3y^2-11+2x}=14+4y-2y^2;$$

$$\therefore\ 3y^2-11+2x+2\sqrt{3y^2-11+2x}=3+4y+y^2;$$

$$\therefore\ (3y^2-11+2x)+2\sqrt{3y^2-11+2x}+1=4+4y+y^2;$$

$$\therefore\ \sqrt{3y^2-11+2x}+1=2+y;$$

$$\therefore\ \sqrt{3y^2-11+2x}=1+y;$$

$$\therefore\ 3y^2-11+2x=1+2y+y^2;$$

$$\therefore\ 2y^2+2x=12+2y;$$

$$\therefore\ y^2+x=6+y;$$

$$\therefore\ y^2+x-y=6.$$

By equation (2.), $y^2=\dfrac{2(x+y)}{x-y}-(x-y);$

$$\therefore\ y^2+x-y=\frac{2(x+y)}{x-y}.$$

$$\text{But } y^2+x-y=6;$$
$$\therefore \frac{2(x+y)}{x-y}=6, \frac{x+y}{x-y}=3;$$
$$\therefore x+y=3x-3y;$$
$$\therefore 2x=4y, x=2y.$$

Substituting, $y^2+2y-y=6$,

$$y^2+y=6,$$
$$y^2+y-6=0;$$
$$\therefore (y+3)(y-2)=0;$$
$$\therefore y=2 \text{ or } -3;$$
$$\therefore x=4 \text{ or } -6.$$

Ex. 18. Find x and y when $x^{\frac{1}{2}}+y^{\frac{1}{2}}=5$. . (1.)

$$xy^{-1}+x^{-1}y+4x^{-\frac{1}{2}}y^{\frac{1}{2}}+4x^{\frac{1}{2}}y^{-\frac{1}{2}}=11\tfrac{13}{36}. \quad . \quad (2.)$$

The second equation is

$$\frac{x}{y}+\frac{y}{x}+4\left(\frac{\sqrt{y}}{\sqrt{x}}+\frac{\sqrt{x}}{\sqrt{y}}\right)=11\tfrac{13}{36};$$
$$\therefore \frac{x}{y}+2+\frac{y}{x}+4\left(\sqrt{\frac{y}{x}}+\sqrt{\frac{x}{y}}\right)=13\tfrac{13}{36};$$
$$\therefore \left(\sqrt{\frac{y}{x}}+\sqrt{\frac{x}{y}}\right)^2+4\left(\sqrt{\frac{y}{x}}+\sqrt{\frac{x}{y}}\right)+4=17\tfrac{13}{36}$$
$$=\frac{625}{36};$$
$$\therefore \sqrt{\frac{y}{x}}+\sqrt{\frac{x}{y}}+2=\frac{25}{6};$$
$$\therefore \sqrt{\frac{y}{x}}+\sqrt{\frac{x}{y}}=\frac{13}{6};$$
$$\therefore \frac{x+y}{\sqrt{xy}}=\frac{13}{6}.$$

From (1.) by squaring, $x+y+2\sqrt{xy}=25$;

$$\therefore x+y=25-2\sqrt{xy};$$
$$\therefore \text{by substitution, } \frac{25-2\sqrt{xy}}{\sqrt{xy}}=\frac{13}{6};$$
$$\therefore 150-12\sqrt{xy}=13\sqrt{xy};$$
$$\therefore 25\sqrt{xy}=150;$$
$$\therefore \sqrt{xy}=6;$$
$$\therefore x+y=13, \text{ and } xy=36,$$

from which x and y may easily be found.

When more than two unknown quantities are given, they may either be reduced to two by substitution, or arranged in some way so that they can be simplified before the number of unknown quantities are reduced, as in the following examples:—

Ex. 19. Find x, y, and z from the equations—

$$x^2+xy+xz=20. \quad . \quad . \quad . \quad (1.)$$
$$y^2+xy+yz=30. \quad . \quad . \quad . \quad (2.)$$
$$z^2+yz+xz=50. \quad . \quad . \quad . \quad (3.)$$

Adding, $x^2+y^2+z^2+2xy+2xz+2yz=100.$

The left side of this equation $=(x+y+z)^2$;

$$\therefore\ x+y+z=\pm 10.$$

Equation (1.) may be written $x(x+y+z)=20$;

$$\therefore\ \pm 10x=20, \text{ or } x=\pm 2.$$

Equation (2.) may be written $y(x+y+z)=30$;

$$\therefore\ y=\pm 3.$$

Equation (3.) may be written $z(x+y+z)=50$;

$$\therefore\ z=\pm 5.$$

Ex. 20. Find x, y, and z from the equations—

$$xy=a. \quad . \quad . \quad (1.)$$
$$xz=b. \quad . \quad . \quad (2.)$$
$$yz=c. \quad . \quad . \quad (3.)$$

Multiplying all three together, $x^2y^2z^2=abc$;

$$\therefore\ xyz=\sqrt{abc}.$$

Dividing by equation (1.), $z=\dfrac{\sqrt{abc}}{a}=\sqrt{\dfrac{bc}{a}}.$

" " " (2.), $y=\dfrac{\sqrt{abc}}{b}=\sqrt{\dfrac{ac}{b}}.$

" " " (3.), $x=\dfrac{\sqrt{abc}}{c}=\sqrt{\dfrac{ab}{c}}.$

Ex. 21. Solve the equations—

$$\frac{xyz}{x+y}=4\frac{4}{5}. \quad . \quad . \quad . \quad (1.)$$
$$\frac{xyz}{x+z}=4. \quad . \quad . \quad . \quad (2.)$$
$$\frac{xyz}{y+z}=3\frac{3}{7}. \quad . \quad . \quad . \quad (3.)$$

Inverting each equation and adding,

$$\frac{2x+2y+2z}{xyz}=\frac{18}{24};$$
$$\therefore\ \frac{x+y+z}{xyz}=\frac{9}{24}.$$

Subtracting reciprocal of equation (1.),

$$\frac{z}{xyz}=\frac{9}{24}-\frac{5}{4}=\frac{1}{6};$$

$$\therefore\ \frac{1}{xy}=\frac{1}{6};$$

$$\therefore\ xy=6.$$

Subtracting reciprocal of equation (2.),

$$\frac{y}{xyz}=\frac{9}{24}-\frac{1}{4}=\frac{3}{24}=\frac{1}{8};$$

$$\therefore\ \frac{1}{xz}=\frac{1}{8};$$

$$\therefore\ xz=8.$$

Subtracting reciprocal of equation (3.),

$$\frac{x}{xyz}=\frac{2}{24}=\frac{1}{12};$$

$$\therefore\ yz=12.$$

These results, $xy=6$, $xz=8$, $yz=12$, may be solved like Ex. 18.

Ex. 22. Solve the equations—

$$x+y+z=14. \quad \ldots \quad (1.)$$
$$x^2+y^2+z^2=84. \quad \ldots \quad (2.)$$
$$xz=y^2. \quad \ldots \quad (3.)$$

By equation (3.), $y^2=xz$, and $y=\sqrt{xz}$.

Substitute the former in equation (2.), the latter in equation (1.);

$$\therefore\ x+\sqrt{xz}+z=14; \quad \ldots \quad (4.)$$
$$x^2+xz+z^2=84. \quad \ldots \quad (5.)$$

Divide (5.) by (4.); $\therefore\ x-\sqrt{xz}+z=\frac{84}{14}=6.$

Add this and (4.) together, and divide by 2;

$$x+z=10. \quad \ldots \quad (6.)$$

Subtracting, $2\sqrt{xz}=8. \quad \ldots \quad (7.)$

Squaring (6.), $x^2+2xz+z^2=100.$

Multiplying (7.) by -1, after squaring,

$$-4xz=-64.$$

Adding, $(x-z)^2=36;$

$$\therefore\ x-z=\pm 6;$$

$$\therefore\ x=8 \text{ or } 2,\ z=2 \text{ or } 8;\ \therefore\ y=\pm 4.$$

Ex. 23. Solve the equations—

$$x+y+z=23. \quad \ldots \quad (1.)$$
$$xy+xz+yz=167. \quad \ldots \quad (2.)$$
$$xyz=385. \quad \ldots \quad (3.)$$

Suppose we have three quantities, $(p-x)$, $(p-y)$, $(p-z)$; their product is

$$p^3-p^2(x+y+z)+p(xy+xz+yz)-xyz.$$

Now, whatever be the values of x, y, and z in the given equations, suppose these values to be substituted for x, y, z respectively in the last expression.

The expression becomes,

$$p^3-23p^2+167p-385.$$

Put this equal to zero;

$$\therefore p^3-23p^2+167p-385=0.$$

Now, since the values of x, y, z are substituted for x, y, z in this expression, and $p-x$, $p-y$, $p-z$ were factors of the expression before substituting the values of x, y, z, the values of x, y, z will now take the place of x, y, z in the factors.

But $p^3-23p^2+167p-385=0=(p-5)(p-7)(p-11)=0$;

$$\therefore x=5, y=7, z=11.$$

And from the nature of the question, each letter x, y, z may have any one of the three values, 5, 7, 11, providing that the other two values are given to the other two letters;

$$\therefore x=5, 7, \text{ or } 11;$$
$$y=11, 5, \text{ or } 7;$$
$$z=7, 11, \text{ or } 5.$$

Ex. 24. Solve $x(y+z)^2=9$. . . . (1.)

$$x+y-z=\frac{3}{2}. \quad . \quad . \quad (2.)$$

$$yz=\frac{3}{16}. \quad . \quad . \quad . \quad (3.)$$

From (1.), $(y+z)^2=\dfrac{9}{x}$;

$$\therefore (y-z)^2=\frac{9}{x}-4yz.$$

From (3.), $4yz=\dfrac{3}{4}$;

$$\therefore (y-z)^2=\frac{9}{x}-\frac{3}{4}.$$

From (2.), $y-z=\dfrac{3}{2}-x$;

$$\therefore (y-z)^2=\frac{9}{4}-3x+x^2;$$

$$\therefore \frac{9}{4}-3x+x^2=\frac{9}{x}-\frac{3}{4};$$

$$\therefore 3-3x+x^2=\frac{9}{x};$$

$$\therefore 3x-3x^2+x^3=9;$$

$$\therefore x^3-3x^2+3x-1=8;$$

$$\therefore (x-1)^3=2^3;$$

$$\therefore x-1=2; \therefore x=3.$$

$$\text{But } (y+z)^2=\frac{9}{x};$$

$$\therefore (y+z)^2=3;$$

$$\therefore y+z=\sqrt{3},$$

$$\text{and } y-z=\frac{3}{2}-x=\frac{3}{2}-3=-\frac{3}{2};$$

$$\therefore 2y=\sqrt{3}-\frac{3}{2}=\frac{2\sqrt{3}-3}{2};$$

$$\therefore y=\frac{1}{4}(2\sqrt{3}-3)=\frac{\sqrt{3}}{4}(2-\sqrt{3});$$

$$2z=\sqrt{3}+\frac{3}{2}; \therefore z=\frac{\sqrt{3}}{4}(2+\sqrt{3}).$$

Ex. 25. Find two numbers whose product shall be equal to the difference of their squares, and the sum of their squares = difference of their cubes.

Let x and vx be the two numbers.

Then by the question,

$$vx^2=v^2x^2-x^2. \quad . \quad . \quad (1.)$$

$$v^2x^2+x^2=v^3x^3-x^3. \quad . \quad . \quad (2.)$$

$$\text{From (1.), } v=v^2-1;$$

$$\therefore v^2-v=1;$$

$$v^2-v+\frac{1}{4}=\frac{5}{4};$$

$$v-\frac{1}{2}=\pm\frac{1}{2}\sqrt{5};$$

$$\therefore v=\frac{1}{2}(1\pm\sqrt{5}).$$

$$\text{From (2.), } v^2+1=v^3x-x;$$

$$\therefore x=\frac{v^2+1}{v^3-1}.$$

$$\text{But } v^2=v+1;$$

$\therefore$ the numerator of this fraction becomes $v+2$.

$$\text{And } v^3 - 1 = (v-1)(v^2+v+1)$$
$$= (v-1)(v+1+v+1) = 2(v^2-1) = 2v;$$
$$\therefore x = \frac{v+2}{2v} = \frac{\frac{1}{2}(1 \pm \sqrt{5}) + 2}{(1 \pm \sqrt{5})} = \frac{1}{2}\sqrt{5};$$
$$\therefore vx = \frac{1}{2}\sqrt{5} \times \frac{1}{2}(1 \pm \sqrt{5}) = \frac{1}{4}(5 + \sqrt{5}).$$

Ex. 26. Out of 126 feet of deal three cubical boxes are to be made, of different sizes, to contain altogether 73 cubic feet. If the sum of their depths is 7 feet, find the depth of each box.

Let x, y, z be the depths of the boxes.

$$\text{Then } x+y+z=7. \quad . \quad . \quad . \qquad (1.)$$
$$6(x^2+y^2+z^2)=126, \text{ or } x^2+y^2+z^2=21. \quad . \qquad (2.)$$
$$x^3+y^3+z^3=73. \quad . \quad . \quad . \qquad (3.)$$

Squaring equation (1.),

$$x^2+y^2+z^2+2xy+2xz+2yz=49.$$

Substituting from (2.),

$$21+2(xy+xz+yz)=49;$$
$$\therefore xy+xz+yz=14 \quad . \quad . \quad . \qquad (4.)$$

Again, cubing equation (1.),

$$(x^3+3x^2y+3xy^2+y^3)+3(x+y)^2z+3(x+y)z^2$$
$$+z^3=343,$$
$$\text{or } x^3+y^3+z^3+3x^2y+3xy^2+3z(x+y)$$
$$(x+y+z)=343.$$

Substituting for $x^3+y^3+z^3$, and for $x+y+z$,

$$73+3x^2y+3xy^2+3z(x+y)\times 7=343,$$
$$\text{or } 3x^2y+3xy^2+21z(x+y)=270;$$
$$\therefore x^2y+xy^2+7z(x+y)=90;$$
$$\therefore x^2y+xy^2+7xz+7yz=90.$$
$$\text{But by } 7xy+7yz+7xz=98.$$
$$\text{Subtracting, } x^2y+xy^2-7xy=-8;$$
$$\therefore xy(x+y-7)=-8.$$
$$\text{But by (1.), } x+y-7=-z;$$
$$\therefore xyz=8.$$

This result is like the equation given in Ex. 23.

$$\text{Hence we get } p^3-7p^2+14p-8=0,$$
$$\text{or } (p-1)(p-2)(p-4)=0;$$

$\therefore$ the depths of the boxes are 1, 2, and 4 feet.

Ex. 27. Solve the equations,

$$x^m y^n = a. \quad . \quad . \quad . \quad . \qquad (1.)$$
$$x^p y^q = b. \quad . \quad . \quad . \quad . \qquad (2.)$$

Raising equation (1.) to the pth power, and equation (2.) to the mth power,

$$x^{mp}y^{np}=a^p,$$
$$x^{mp}y^{mq}=b^m;$$
$$\therefore \frac{y^{np}}{y^{mq}}=\frac{a^p}{b^m};$$
$$\therefore y^{np-mq}=\frac{a^p}{b^m};$$
$$\therefore y=\left(\frac{a^p}{b^m}\right)^{\frac{1}{np-mq}}.$$

If the unknown quantities are exponents of known quantities, it is generally necessary to use logarithms in their solutions. We therefore give a few examples of equations involving the use of logarithms.

Ex. 28. Having given log $2 = \cdot 301030$, and log $3=\cdot 477121$, find the value of x which satisfies the equation $2^x=729$.

$$729=3^6;\ \therefore \log 729=6\times\log 3=2\cdot 862726.$$

But if $2^x=729$, $x \log 2=\log 729=6 \log 3$;

$$\therefore x=\frac{6 \log 3}{\log 2}=\frac{2\cdot 862726}{\cdot 301030}$$
$$=9\cdot 5.$$

Ex. 29. Having log $2 = \cdot 301030$, and log $109 = 2\cdot 0374265$, find x from the equation

$$\left(\frac{5}{4}\right)^x=54\frac{1}{2}.$$

$$\text{Log } 4=2 \log 2=2\times\cdot 301010=\cdot 602020,$$
$$\log 5=\log 10-\log 2=1-\cdot 301010=\cdot 698990.$$

$$\text{But if } \left(\frac{5}{4}\right)^n=54\tfrac{1}{2}=\frac{109}{2},$$
$$x(\log 5-\log 4)=\log 109-\log 2;$$
$$\therefore x=\frac{\log 109-\log 2}{\log 5-2 \log 2}$$
$$=\frac{2\cdot 037426-\cdot 301010}{\cdot 698990-\cdot 602020}=17\cdot 17.$$

Ex. 30. Find x and y when

$$a^{(x+y)^2}=b^{(x+y)^3}, \text{ and } a_1^{(x-y)^2}=b_1^{(x-y)^3}.$$

The first gives $(x+y)^2 \log a = (x+y)^3 \log b$;

$$\therefore \log a = (x+y) \log b;$$

$$\therefore x+y = \frac{\log a}{\log b}.$$

The second gives $(x-y)^2 \log a_1 = (x-y)^3 \log b_1$;

$$\therefore \log a_1 = (x-y) \log b_1;$$

$$\therefore x-y = \frac{\log a_1}{\log b_1}.$$

Adding the two results together,

$$2x = \frac{\log a}{\log b} + \frac{\log a_1}{\log b_1};$$

$$\therefore x = \tfrac{1}{2}\left\{\frac{\log a}{\log b} + \frac{\log a_1}{\log b_1}\right\}.$$

Subtracting, etc.,

$$y = \tfrac{1}{2}\left\{\frac{\log a}{\log b} - \frac{\log a_1}{\log b_1}\right\}.$$

Ex. 31. Find x when

$$(a^4 - 2a^2b^2 + b^4)^{x-1} = \frac{(a-b)^{2x}}{(a+b)^2}.$$

Here $(x-1) \log (a^4 - 2a^2b^2 + b^4) = 2x \log (a-b)$
$- 2 \log (a+b)$,

or $2(x-1) \log (a^2 - b^2) = 2x \log (a-b) - 2 \log (a+b)$,

or $2(x-1) \log (a-b) + 2(x-1) \log (a+b) = 2x \log (a-b) - 2 \log (a+b)$;

$$\therefore \log (a-b)\{2(x-1) - 2x\} + \log (a+b)\{2(x-1)+2\} = 0;$$

$$\therefore \log (a-b)\{-2\} + \log (a+b)\{2x\} = 0;$$

$$\therefore 2x \log (a+b) = 2 \log (a-b);$$

$$\therefore x \log (a+b) = \log (a-b);$$

$$\therefore x = \frac{\log (a-b)}{\log (a+b)}.$$

Ex. 32. Given $(\sqrt[r]{a^c})^x = b^{cx-4a}$, to find x.

Here $(a^c)^{\frac{x}{r}} = b^{cx-4a}$;

$$\therefore (a)^{\frac{cx}{r}} = b^{cx-4a};$$

$$\therefore \frac{cx}{r} \log a = (cx - 4a) \log b;$$

$$\therefore \frac{cx}{r} (\log a - r \log b) = -4a + \log b;$$

$$\therefore x = \frac{4ar \log b}{c(\log b - \log a)}.$$

Ex. 33. Find x when $3^{2x} . 5^{3x-4} = 7^{x-1} . 11^{2-x}$.

$$2x \log 3 + (3x-4) \log 5 = (x-1) \log 7 + (2-x) \log 11.$$

$$x\,(2 \log 3 + 3 \log 5 - \log 7 + \log 11) = 4 \log 5 - \log 7 + 2 \log 11;$$

$$\therefore\ x = \frac{4 \log 5 - \log 7 + 2 \log 11}{2 \log 3 + 3 \log 5 - \log 7 + \log 11}.$$

Sometimes this kind of equations can be solved without the use of logs, as in Ex. 14 in this chapter. The following are additional examples :—

Ex. 34. Find x when $2^{2^x} + 1 = 257$.

Let $y = 2^x$; $\therefore\ 2^y = 256$.

Arranging 256 as a power of 2, we get

$$2^y = 2^8;\ \therefore\ y = 8.$$

$$\text{But } 2^x = y;\ \therefore\ 2^x = 8 = 2^3;$$

$$\therefore\ x = 3.$$

Ex. 35. Find x and y, when $y^{x+y} = x^{4a}$. . (1.)

$x^{x+y} = y^a$. . (2.)

$$\text{From (1.), } y = x^{\frac{4a}{x+y}}.$$

$$\text{From (2.), } y = x^{\frac{x+y}{a}}.$$

Equating the indices,

$$\frac{4a}{x+y} = \frac{x+y}{a};$$

$$\therefore\ (x+y)^2 = 4a^2;$$

$$\therefore\ x+y = 2a;$$

$$\therefore \text{ by substitution, } y^{2a} = x^{4a};$$

$$\therefore\ y = x^2.$$

$$\text{But } x+y = 2a;$$

$$\therefore\ x^2 + x = 2a,$$

from which the value of x and y may be found.

The following example was given at the Science Examination, May 1878 :—

Ex. 36. Solve

$$\frac{3^{5x} - 2^{5x}}{3^{5x} + 2^{5x}} = \frac{19}{35}, \text{ and } \frac{\log 3x + \log 2x}{\log 3x - \log 2x} = \frac{5}{3}.$$

If $\dfrac{3^{5x} - 2^{5x}}{3^{5x} + 2^{5x}} = \dfrac{19}{35}$, $\dfrac{2.3^{5x}}{2.2^{5x}} = \dfrac{54}{16}$; $\therefore\ \dfrac{3^{5x}}{2^{5x}} = \dfrac{27}{8}$;

$$\therefore\ 5x \log 3 - 5x \log 2 = \log 27 - \log 8 = \log (3^3) - \log (2^3) = 3 \log 3 - 3 \log 2;$$

$$\therefore\ 5x = 3;\ \therefore\ x = \frac{3}{5}.$$

If $\frac{\log 3x+\log 2x}{\log 3x-\log 2x}=\frac{5}{3}$, then $\frac{2 \log 3x}{2 \log 2x}=\frac{8}{2}=4$;

$$\therefore \log 3x=4 \log 2x.$$

Now the numbers $3x$, $2x$ are in the proportion $3:2$, and their logs are as $4:1$.

For convenience we will take y and z as the numbers.

Then $y:z::3:2$, and $\log y:\log z::4:1$;

$$\therefore 2y=3z, \text{ and } \log y=4 \log z.$$

From the first, $\log 2+\log y=\log 3+\log z$;

$$\therefore \log 2+4 \log z=\log 3+\log z;$$

$$\therefore 3 \log z=\log 3-\log 2,$$

$$\log z=\frac{\log 3-\log 2}{3};$$

$$\therefore \log 2x=\frac{\log 3-\log 2}{3},$$

$$\text{and } \log 3x=\frac{4(\log 3-\log 2)}{3},$$

from either of which $\log x$ may be found.

Before concluding the subject of equations, we may point out that there must always be as many equations as there are unknown quantities; and these equations must be independent of each other—that is, no one of them must be capable of being formed by any arrangement or change of the others. It is sometimes difficult to tell at first sight whether the given equations are totally independent of each other or not, as in the following example:—

$$x+2y+3z=10,$$
$$2x-y+z=5,$$
$$3x+11y+14z=45.$$

At first sight these appear to be independent of each other, but such is not the case, for if the first be multiplied by 5, and the second subtracted from it, the result will be an equation identical with the third equation. The result is, that we have only two independent equations, and three unknown quantities; so that, although $x=3$, $y=2$, $z=1$ will satisfy them all, there are other values of x, y, and z which will satisfy them, and the equations are indeterminate.

Sometimes, also, we arrive at a curious result in an equation of one unknown quantity. For example,

$$\text{Let } \frac{x+1}{x+3}=\frac{x+2}{x+4};$$

that is, $(x+1)\ (x+4)=(x+2)\ (x+3)$, or $x^2+5x+4 =x^2+5x+6$. The unknown quantity now entirely disappears, and the only value of x which can satisfy the equality is $x=$ infinity.

Considering $\frac{x+1}{x+3}$ and $\frac{x+2}{x+4}$ as two ratios, the second is derived from the first by adding unity to each of its parts, which we know necessarily alters the value of the ratio. It cannot, therefore, be equated with what it was before this alteration; consequently, $\frac{x+1}{x+3}$ cannot be equated to $\frac{x+2}{x+4}$ if x has any finite value.

III. GEOMETRY.

THE subjects of this section are arranged under five heads:—

(*a*) The relations and properties of similar rectilinear figures.

(*b*) The elementary properties of the plane, including those of the angles made by planes with right lines and with each other.

(*c*) The elementary properties of the sphere, including those of great and small circles on the surfaces ot spheres.

(*d*) The mensuration of the simpler plane and solid figures, including the circle, the sphere, the cylinder, and the cone.

(*e*) Elements of co-ordinate geometry in rectangular and polar co-ordinates, as far as the equations and properties of the right line and circle.

(*a*) THE RELATIONS AND PROPERTIES OF SIMILAR RECTILINEAR FIGURES.

The sixth book of Euclid will be sufficient for this branch of geometry. Before beginning it, the student will do well to read carefully the definitions of Book v., so as to obtain a clear knowledge of proportional parts

as applied to figures by Euclid. The best editions of Euclid are those of Longmans, by Mr. Potts; and Phillips, by Mr. J. Martin. In these editions, the arrangement of the steps in the proofs is an assistance in remembering the propositions, and the latter has the additional advantage of having the figures repeated when the proof of the proposition is continued over a leaf, thus saving the trouble of turning back to refer to the diagrams.

It is hardly necessary to remind the student that a mere verbal knowledge of the propositions is not sufficient. He must practise *deductions* on the propositions, and should be able, in working out these deductions, to refer to the number of the book, and the number of the proposition which he uses in his demonstrations. A copious selection of deductions will be found in either of the editions of Euclid above referred to.

Algebra may sometimes be applied with advantage in working out deductions on Euclid, but it must not be used in the demonstrations. By means of algebra we may find out how to construct the figures and prepare the proposition for the application of Euclid; but this must be the limit of its use, unless an algebraic proof is specially demanded. This application of algebra is shown in the following examples:—

Ex. 1. To produce a given line, so that the square on the whole line produced may be twice the square on the given line.

Suppose the given line to contain a units of length.

Let it be produced x units to meet the conditions of the question.

Then the whole line produced will be $a+x$ units. By the question,

$$2a^2=(a+x)^2; \therefore a+x=a\sqrt{2};$$
$$\therefore x=a\sqrt{2}-a.$$

Now if we describe a square upon a line containing a units, the diagonal of the square will contain $a\sqrt{2}$ units;

$$\therefore a\sqrt{2}-a=\text{diagonal of square}-\text{side}.$$

If, therefore, we produce one side of the square, and make the side produced equal to the diagonal, the square on the line thus produced will be twice the square of the given line itself.

The proof of this may be thus expressed :—

Let AB be the given line.

It is required to produce AB to the point C, so that the square on AC shall be twice the square on AB.

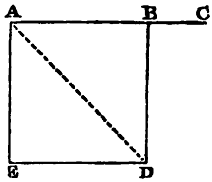

Fig. 1.

On AB describe a square ABDE. Draw the diagonal AD.

Produce AB to C, making AC = AD.

Then the square on AC shall be equal to twice the square on AB.

Because ABDE is a square, AB = BD, and ABD is a right angle;

$$\therefore AB^2 + BD^2 = AD^2 \text{ (I. 47)};$$
$$\therefore AD^2 = 2AB^2.$$
$$\text{But } AD = AC; \therefore AC^2 = 2AB^2;$$

∴ the line AB has been produced to C, so that the square of the whole line AC is twice the square of AB. (Q. E. F.)

Ex. 2. Divide a given straight line into two parts, so that the difference of the squares of the two parts may be equal to twice the rectangle contained by the two parts.

Let the given line AB contain a units.

Let x and $a-x$ be the two parts.

Then by the question,

$$x^2 - (a-x)^2 = 2x(a-x);$$
$$\therefore x^2 - a^2 + 2ax - x^2 = 2ax - 2x^2;$$
$$\therefore a^2 = 2x^2,$$
$$x = \frac{a}{\sqrt{2}} = \frac{a\sqrt{2}}{2}.$$

But $\frac{a\sqrt{2}}{2}$ will be half the diagonal of the square on AB;

∴ one of the parts required will be half the length of the diagonal.

This gives us the data for the problem, and we may prove it by Euclid, thus:

Let AB be the given line.

On AB describe a square, ABCD. Draw the diagonal AC. Bisect AC in E. Make $AF = AE$. Draw FGM parallel to AD or BC, and KGH parallel to AB or DC.

Then the difference between the squares on AF and BF shall be equal to twice the rectangle BF . AF.

FK is the square on AF, HM the square on BF. FH, KM are each equal to the rectangle BF . FA.

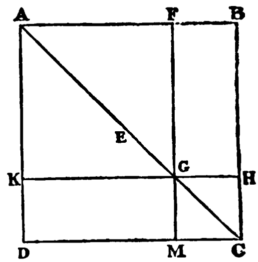

Fig. 2.

$$\text{Now } AB^2 + BC^2 = AC^2 \text{ (I. 47).}$$
$$\text{But } AB^2 = AF^2 + BF^2 + 2AF \,.\, FB \text{ (II. 4).}$$
$$\text{Similarly, } BC^2 = BH^2 + CH^2 + 2BH \,.\, CH.$$
$$\text{But } AF = BH, \text{ and } BF = CH;$$

$\therefore$ by addition,

$$AB^2 + BC^2 = 2AF^2 + 2BF^2 + 4AF \,.\, FB$$
$$= 2(AF^2 + BF^2 + 2AF \,.\, BF).$$
$$\text{Also } AC^2 = 4AE^2, \text{ and } AE = AF;$$
$$\therefore AC^2 = 4AF^2;$$
$$\therefore 4AF^2 = 2(AF^2 + BF^2 + 2AF \,.\, BF);$$
$$\therefore 2AF^2 = AF^2 + BF^2 + 2AF \,.\, BF;$$
$$\therefore AF^2 = BF^2 + 2AF \,.\, BF;$$
$$\therefore AF^2 - BF^2 = 2AF \,.\, BF.$$

Wherefore the line AB has been divided, so that the difference of the squares of the two parts is equal to twice the rectangle contained by the two parts.

By means of Euclid we may construct lines which shall have a ratio involving surd quantities; thus :—

Ex. 3. Having given a line as unity, construct three lines which shall be to each other as $\sqrt{10} : \sqrt{2} : \sqrt{5}$.

Let AB be the given line.

On AB describe a square ABCD. Join AC.

Then

$$AC^2 = BC^2 + AB^2$$
$$= 2AB^2;$$
$$\therefore AC = AB\sqrt{2}$$

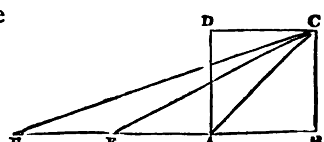

Fig. 3.

Produce BA to E. Make $AE = AB$. Join EC.

$$\text{Then } EB^2+BC^2=EC^2.$$
$$\text{But } EB=2AB, \text{ and } BC=AB;$$
$$\therefore 4AB^2+AB^2=EC^2;$$
$$\therefore 5AB^2=EC^2;$$
$$\therefore EC=AB\sqrt{5}.$$

Again, produce AE to F, making EF = AB. Join CF.

$$\text{Then } BF=3AB;$$
$$\therefore BF^2=9AB^2,$$
$$\text{and } BF^2+BC^2=CF^2;$$
$$\therefore 9AB^2+AB^2=CF^2;$$
$$\therefore CF^2=10AB^2;$$
$$\therefore CF=AB\sqrt{10};$$

$\therefore$ the three lines, CF, AC, EC, are in the ratio of $\sqrt{10} : \sqrt{2} : \sqrt{5}$.

Ex. 4. Find the ratio of the diagonal of a square to the diagonal of a cube, of which the square is one of its faces.

Let ABCD be the square forming one face of the cube, AC the diagonal of the square, and CE the diagonal of the cube.

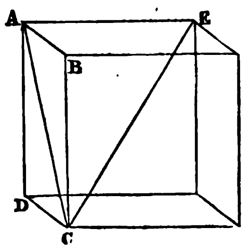

Fig. 4.

Then

$$AC^2=AB^2+BC^2=2AB^2;$$
$$\therefore AC=AB\sqrt{2}.$$

The angle CAE is a right angle;

$$\therefore CE^2=CA^2+AE^2$$
$$=CA^2+AB^2$$
$$=2AB^2+AB^2=3AB^2;$$
$$\therefore CE=AB\sqrt{3};$$
$$\therefore AC : CE :: \sqrt{2} : \sqrt{3}.$$

In applying algebra to the solution of deductions, these methods of finding ratios are frequently of great value.

The second book of Euclid is the one to which algebra can be applied most readily, but the laws of proportion in algebra are almost equally applicable to solutions of deductions on Book VI.

Another way of discovering the mode of proof in the case of deductions, is to consider the required result to be true, and from its truth to discover some peculiarity on which the proof depends, which is set forth in Euclid. We give an example of this method.

Ex. 5. Divide the circumference of a circle into two parts, so that the angle in one segment may be twice that in the other.

Let ABCD be the circle. Suppose the circle divided into two segments so that the angle ABC is double the angle ADC. Draw the diameter BD. Join AD, CD, BC, AB, AC.

The angle DAB is a right angle, for it is the angle in a semicircle, and the angles ADB, ABD are together equal to one right angle.

But angle ABD is double ADB;

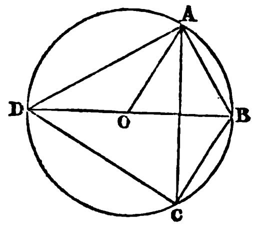

Fig. 5.

∴ angle ABD is two-thirds of a right angle—that is, it is the angle of an equilateral triangle.

This gives us a clue to the method of solution as required by the question, which may be thus stated :—

Draw any diameter BD.

Let O be the centre of the circle. On BO describe an equilateral triangle, as in Euclid (I. 1).

Let A be the intersections of the circles, or vertex of the triangle. Join AD. Draw AC perpendicular to BO, cutting the circumference in C on the other side of the diameter. Join CD, CB.

Then the segment ABC shall contain an angle which is double of the angle in the segment ADC.

Because DAB is a semicircle,

∴. angle DAB is a right angle;

∴. angles ADB, ABD are together equal to a right angle.

But angle ABD is angle of an equilateral triangle, or two-thirds of a right angle;

∴. angle ADB is one-third of a right angle; wherefore angle ABD is double angle ADB.

For similar reasons, angle CBD is double CDB;

∴. angle ABC is double of angle ADC;

∴. the segment ABC contains an angle which is double of the angle contained in the segment ADC. (Q. E. F.)

The method of indirect proof is sometimes used in working deductions, as in the propositions of Euclid, but if a direct proof can be given, it is best to give one.

Exercises are sometimes given which require other principles to be applied to them besides those of Euclid; the following is an example of this kind :—

Ex. 6. ABC is a right-angled triangle, having a semicircle described on each of its sides.

Show that the lunes formed by the circumference of the semicircle on the hypotenuse, and the circumferences of the smaller circles, are together equal to the right-angled triangle.

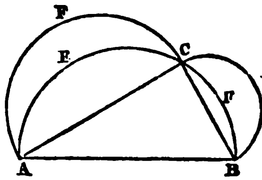

Fig. 6.

Let C be the right angle.

Because the triangle ABC is right-angled, therefore the squares on AC and BC are together equal to the square on AB.

But circles and semicircles are to each other as the squares of their diameters;

∴ the semicircles on AC and BC are equal to the semicircle on AB.

But the semicircle on AB is made up of the segment AEC, the segment BFC, and the right-angled triangle ABC;

∴ segment AEC + segment BFC + triangle ABC = semicircle AFC + semicircle BDC.

Take from each the common parts, viz. the segments AEC and BFC;

∴ the right-angled triangle ABC is equal to the lune contained by AFC, AEC, and that contained by BDC and BFC.

There are many geometrical problems which are deductions from Euclid's elements that are more easily proved by the application of plane co-ordinate geometry. After studying this subject, the student will find his power of working out geometrical propositions marvellously increased. Many exercises which would require a considerable time to write out, may be done in a few lines when the principles of co-ordinate geometry are applied to them. This remark applies more especially to loci, points of intersection, and properties of the circle.

(*b*) The Elementary Properties of the Plane, etc.

After finishing the sixth book of Euclid, the student should proceed to Book XI. and get up Props. 1–21, working out exercises as advised in studying Book VI.

Book XI. is concerned chiefly with solid angles—that is, with angles made by planes. There are also a few propositions which are not given in the eleventh book of Euclid which the student must keep in mind, as they are frequently required in the mensuration of solids. They are as follows:—

1. Similar superficial figures are to each other as the squares of corresponding lines. For example, circles are to each other as the squares of their diameters, or the squares of chords on which equal angles stand.

2. Similar solids are to each other as the cubes of corresponding lines.

3. Every sphere or hemisphere is two-thirds of its circumscribing cylinder.

4. Every cone is one-third of its circumscribing cylinder.

(*c*) The Elementary Properties of the Sphere, including those of Great and Small Circles on the Surface of Spheres.

The following are the chief points included in this subject:—

1. Every section of a sphere made by a plane is a circle.

Let ABCD be a sphere of which P is the centre.

Let AECF be the curve forming the surface where the plane cuts it.

Draw PB perpendicular to this plane, and produce it till it meets the surface of the sphere at D.

Join DE, EB. Draw EK perpendicular to PB, and join EP.

Now, since PK, BK are each perpendicular to the plane AECF, they are perpendicular to every line in that plane which they meet at K;

$\therefore$ BKE is a right angle; and $KE^2 = EB^2 - BK^2$;

$\therefore EK = \sqrt{EB^2 - BK^2}$.

Now, E is any point in AECF;

$\therefore \sqrt{EB^2 - BK^2}$ is a constant quantity;

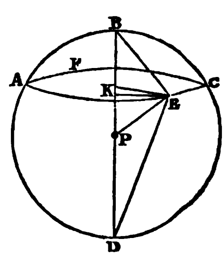

Fig. 7.

∴ any point in AECF is equidistant from K;
∴ AECF is a circle.

2. By a similar method it may be proved that if two spheres intersect, the part common to both their surfaces is a circle.

3. If a plane which cuts a sphere passes through the centre of the sphere, the circle forming the section is called a great circle. If it does not pass through the centre, it is called a small circle. In the above example, AECF is a small circle.

4. Since all great circles have their radii equal to the radius of the sphere, all great circles on the same sphere are equal; and the radii of small circles decrease as the centre of the circle recedes from the centre of the sphere.

5. If the centre of the sphere be joined with the centre of a small circle, or a line be drawn perpendicular to the plane of a great circle from the centre of the sphere, the point where this line meets the surface of the sphere is called the pole. A clear idea of this may be obtained by reference to a terrestrial globe. The equator and the ecliptic are great circles, the arctic and antarctic circles are small circles, and the North and South Poles are the ends of a line perpendicular to the plane of the equator and the small circles referred to.

6. The pole of a circle, whether a great or small circle, is equidistant from every point in its circumference. By reference to the figure of Article 1, this proposition is easily proved.

7. The angle between any circle whatever and a great circle passing through its pole is a right angle. This may be illustrated by reference to a terrestrial globe, as the meridians passing through the North and South Poles meet the equator and the parallels of latitude at right angles.

8. If any point be taken in the circumference of a great circle, and a line at right angles to it be drawn on the sphere, the poles of this great circle will be upon this line at distances from the great circle of one-fourth and three-fourths of the circumference of the great circle.

9. If two great circles intersect, they must bisect each other, for two circles can only intersect in two points; and if their centres coincide, it is evident that the points

of intersection will be at the ends of a diameter common to both circles.

10. Any two circles described upon the same sphere are to each other as the squares of the distances of their centres from the nearest of their poles.

For circles are to each other as the squares of their diameter, or as the squares of their radii.

But the radii vary according to their distances from the pole;

∴ the areas of circles are to each other as the squares of their distances from the nearest pole.

11. The area of the lune enclosed between two intersecting great circles depends upon the angle at which the great circles intersect. Thus if the great circles of which AEB, ADB are parts, meet at an angle of $30°$, the area enclosed between AEB and ADB will be $\frac{30}{360}$ of the whole area of the sphere $=\frac{1}{12}$ of total area of the sphere. As an illustration, we may take the intersection of the equator and the ecliptic. These great circles intersect at an angle of $23° 27' 24''$;

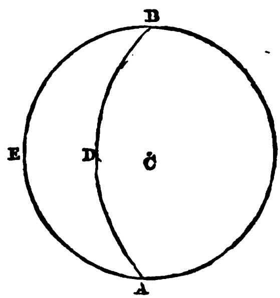

Fig. 8.

∴ the area of the earth's surface included between the two lunes formed by the equator and ecliptic $=2 \cdot \frac{23° \cdot 27' \cdot 24''}{360°}=\frac{23° \cdot 27' \cdot 24''}{180°}$ of whole surface. We shall refer to the mode of finding the surface of the sphere in a subsequent chapter.

(*d*) The Mensuration of the Simpler Plane and Solid Figures, including that of the Circle, the Sphere, the Cylinder, and the Cone.

The best book for mensuration is Professor Elliott's, published by Messrs. W. Stewart & Co. We shall refer to the more important propositions in this work, and advise the reader not to satisfy himselt by merely reading them and committing the formulæ to memory, but to master the proofs of those propositions which do not

equire a knowledge of trigonometry, leaving the others until after trigonometry has been mastered.

Many of the most important rules of mensuration are proved by Euclid's elements. Some require a knowledge of trigonometry, others involve a knowledge of the differential and integral calculus; but in cases where the proof requires a higher knowledge than that given by trigonometry, the pupil may remember the formula, and use it without studying the proof.

The part of the above-mentioned work which the student must commence with, begins at page 106. The first part (to page 157) is concerned with the mensuration of lines, and the important problems for the student for London Examinations are 1, 4, 6, 7, 19, 21.

A very useful constant, called by the Greek letter π, is frequently used in mensuration. It represents the incommensurative quantity which represents the ratio of the circumference of a circle to its diameter. Its value is very nearly 3·14159; but this is a little too small. There are various series for finding its value, but how those series are obtained is rather beyond our subject at present; we shall refer to the proof of one or two of them in a subsequent chapter. The following are series from which the value of π may be calculated:—

$$\frac{\pi}{4}=4\left\{\frac{1}{5}-\frac{1}{3\cdot 5^3}+\frac{1}{5\cdot 5^5}-\text{etc.}\right\}-\left\{\frac{1}{239}-\frac{1}{3}\cdot\frac{1}{239^3}+\frac{1}{5}\cdot\frac{1}{239^5}-\text{etc.}\right\},$$

$$\frac{\pi}{4}=\left(\frac{1}{2}+\frac{1}{3}\right)-\frac{1}{3}\left(\frac{1}{2^3}+\frac{1}{3^3}\right)+\frac{1}{5}\left(\frac{1}{2^5}+\frac{1}{3^5}\right)-\text{etc.},$$

$$\frac{\pi}{6}=\frac{1}{2}+\frac{1}{1\cdot 2}\cdot\frac{1}{3\cdot 8}+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{32\cdot 5}+\text{etc.},$$

$$\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\text{etc.},$$

$$\frac{\pi^2}{6}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\text{etc.}$$

In the mensuration of surfaces, Probs. 1–27 must be mastered.

It will be useful for the student to try to find the rules for the areas of some figures for himself. For example:

Ex. 7. Find the area of a hexagon whose side is a.

Let ABCDEF be the hexagon.

Let O be the centre of the circumscribing circle.

All the angles at O are 60° each;

$\therefore$ in each of the six triangles the angles at the base are 60° each, for OD, OE, etc. are all equal;

$\therefore$ each of the triangles is equilateral.

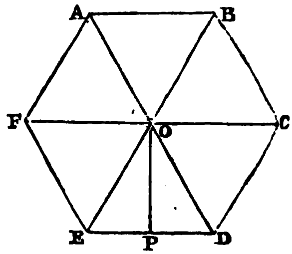

Fig. 9.

Draw OP perpendicular to ED.

Then area of triangle $OED = \dfrac{OP \cdot ED}{2}$.

But OPD is a right-angled triangle;

$$\therefore OD^2 = OP^2 + DP^2$$
$$= OP^2 + (\tfrac{1}{2}OD)^2;$$
$$\therefore OP^2 = \frac{3 \cdot OD^2}{4};$$
$$\therefore OP = \tfrac{1}{2}\sqrt{3} \cdot OD;$$
$$\therefore \text{area of } OED = \frac{\tfrac{1}{2}OD\sqrt{3} \cdot ED}{2} = \tfrac{1}{4}\sqrt{3} \cdot OD^2$$
$$= \frac{a^2}{4}\sqrt{3};$$
$$\therefore \text{area of hexagon} = \frac{6}{4} \cdot \sqrt{3} \cdot a^2 = \frac{3\sqrt{3}}{2}a^2.$$

Ex. 8. To find the area of an octagon whose side is a.

Area = OPQR + 4ABPO + 4AOH.

OP, PQ, QR, OR each = AB = a;

$\therefore$ area OPQR = $AB^2 = a^2$.

AH = AB, and AO = HO;

$\therefore$ 4ABPO = 4AB . AO.

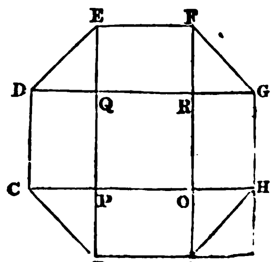

Fig. 10.

$$\text{But } AH^2 = AO^2 + OH^2$$
$$= 2AO^2;$$
$$\therefore AB^2 = 2AO^2;$$
$$\therefore AO = \frac{AB}{\sqrt{2}};$$
$$\therefore 4ABPO = \frac{4 \cdot AB^2}{\sqrt{2}}$$
$$= 2\sqrt{2} \cdot a^2.$$

And,

$$4AOH = \frac{4AO \cdot HO}{2} = \frac{4AO^2}{2} = \frac{4 \cdot \frac{AB^2}{2}}{2} = AB^2 = a^2;$$

$$\therefore \text{ area of octagon} = a^2 + 2a^2\sqrt{2} + a^2$$
$$= 2a^2 + 2a^2\sqrt{2}$$
$$= 2a^2(1 + \sqrt{2}).$$

The problems on the mensuration of solids of greatest importance are 1 to 18 and 47. These must be well studied, after which it will be advantageous for the student to exercise his knowledge by practising the examples at the end of the book.

The following are examples taken from examination papers :—

Ex. 9. An equilateral triangle being supposed to revolve round a line through its vertex perpendicular to the base, show that the area of the cone generated by either of its sides is double that of the circle generated by its base.

Let a be the side of the equilateral triangle;

$$\therefore \text{ diameter of base of cone} = a;$$

$$\therefore \text{ area of base} = \frac{a^2\pi}{4}.$$

Slant side of cone $= a$, and circumference of base $= a\pi$;

$$\therefore \text{ area of slant side} = \tfrac{1}{2}a \cdot a\pi = \frac{\pi a^2}{2};$$

$$\therefore \text{ area of slant side} = \text{twice area of the base.}$$

Ex. 10. Three cylinders of equal altitude being supposed to have for bases the three circles described on the three sides of a right-angled triangle as diameters; show that the volume of the greatest of them is equal to the volumes of the other two together.

The square on the hypotenuse = squares on the two sides;

$\therefore$ circles on hypotenuse=circles on other two sides.

And as the altitudes are equal, and the volumes are one-third the altitude × areas of bases,

$\therefore$ the cylinder having circle on hypotenuse as its base will be equal to the other two.

Ex. 11. A cone and a hemisphere being supposed to have a common base, and to lie on opposite sides of it;

required the ratio of the altitude of the cone to the radius of the hemisphere in order that the volumes of both solids may be equal.

Let x = altitude of cone,
y = radius of hemisphere;

$$\text{volume of cone} = \frac{(2y)^2 \times \frac{\pi}{4} . x}{3} = \frac{4xy^2\pi}{12},$$

$$\text{volume of sphere} = \frac{(2y)^3 \frac{\pi}{6}}{2} = \frac{8y^3\pi}{12};$$

and if the volumes are equal,

$$\frac{4x\pi y^2}{12} = \frac{8y^3\pi}{12};$$

$$\therefore\ 2x = y;$$

∴ altitude of cone must be twice the radius of the sphere.

Ex. 12. Given the radius of a cylinder, find the altitude when the area of the whole surface is four times that of the base.

From this, the area of the curved surface must be twice the area of either end.

Let a = diameter of end or base.

$$\text{Then area of end} = \frac{a^2\pi}{4}.$$

Let x = altitude.

$$\text{Then area of curved surface} = a \times \pi \times x$$
$$= a\pi x;$$

$$\therefore\ \frac{2a^2\pi}{4} = a\pi x,$$
$$a^2\pi = 2a\pi x;$$
$$\therefore\ a = 2x;$$

∴ the altitude must be half the diameter of the base.

Ex. 13. Find the altitude of a cone having the same base as a cylinder such that the area of its curved surface may be equal to that of the curved surface of the cylinder, and determine the ratio of the volumes of the cone and cylinder (the cylinder referred to being similar to the one in the preceding example).

$$\text{Area of curved surface of cylinder} = a\pi x = \frac{a^2\pi}{2}.$$

Let x = altitude of the cone, and a the diameter of the base.

Then area of curved surface

$$=\frac{a.\pi \times \text{slant height}}{2}.$$

$$\text{But slant height}=\sqrt{x^2+\frac{a^2}{4}}=\tfrac{1}{2}\sqrt{4x^2+a^2};$$

$$\therefore \text{ area}=\frac{a\pi}{2.2}\sqrt{4x^2+a^2};$$

$$\therefore \frac{a^2\pi}{2}=\frac{a\pi}{4}\sqrt{4x^2+a^2};$$

$$\therefore 2a=\sqrt{4x^2+a^2};$$

$$\therefore 4a^2=4x^2+a^2;$$

$$\therefore 3a^2=4x^2;$$

$$\therefore x=\frac{a\sqrt{3}}{2},$$

which is the height of the cone.

To compare their solidities,

$$\text{altitude of cone}=\frac{a\sqrt{3}}{2}, \text{ and diameter of base}=a;$$

$$\therefore \text{ solidity}=\frac{a^2\pi}{4}\times\frac{a\sqrt{3}}{2}.$$

Cylinder has $\frac{a}{2}$ for its altitude, and a for diameter of base;

$$\therefore \text{ solidity}=\frac{a^2\pi}{4}\cdot\frac{a}{2};$$

$$\therefore \text{ solidity of cone : solidity of cylinder}$$

$$::\frac{a^2\pi.a\sqrt{3}}{8}:\frac{a^2\pi.a}{8}$$

$$::\sqrt{3}:1.$$

Ex. 14. The sides of a triangle are in arithmetical progression, and its area is four-fifths of that of an equilateral triangle of the same perimeter. Show that the sides of the triangle are as the numbers 7, 10, 13.

Let $x-y$, x, $x+y$ be the three sides. Then the sum of the sides $=3x$;

$$\therefore \text{ side of equilateral triangle}=x;$$

$$\therefore \text{ altitude of equilateral triangle}=\frac{x\sqrt{3}}{2};$$

$$\therefore \text{ area of equilateral triangle} = \frac{x \times x\sqrt{3}}{2 \times 2} \times \frac{4}{5}$$
$$= \frac{x^2\sqrt{3}}{5}.$$

Half sum of sides of other triangle $= \frac{3x}{2}$;
$\therefore$ area

$$= \sqrt{\frac{3x}{2}\left(\frac{3x}{2}-(x-y)\right)\left(\frac{3x}{2}-x\right)\left(\frac{3x}{2}-(x+y)\right)}$$
$$= \sqrt{\frac{3x}{2}\left(\frac{3x-2x+2y}{2}\right)\left(\frac{3x-2x}{2}\right)\left(\frac{3x-2x-2y}{2}\right)}$$
$$= \sqrt{\frac{3x}{2}\left(\frac{x+2y}{2}\right)\left(\frac{x}{2}\right)\left(\frac{x-2y}{2}\right)}$$
$$= \sqrt{\frac{3x^2}{4}\cdot\left(\frac{x^2-4y^2}{4}\right)} = \frac{x^2\sqrt{3}}{5} \text{ by question;}$$
$$\therefore \frac{3x^2}{4}\cdot\frac{x^2-4y^2}{4} = \frac{3x^4}{25};$$
$$\therefore \frac{1}{4}\cdot\frac{x^2-4y^2}{4} = \frac{x^2}{25};$$
$$\therefore \frac{x^2-4y^2}{16} = \frac{x^2}{25};$$
$$\therefore 25x^2 - 100y^2 = 16x^2;$$
$$\therefore 9x^2 = 100y^2;$$
$$\therefore 3x = 10y;$$
$$\therefore x = \frac{10y}{3}.$$

$$\therefore \text{ The sides are as } \frac{10y}{3}-y : \frac{10y}{3} : \frac{10y}{3}+y$$
$$:: \frac{7y}{3} : \frac{10y}{3} : \frac{13y}{3}$$
$$:: 7 : 10 : 13.$$

Ex. 15. A sphere is cut by two parallel planes. One cuts off a segment p inches high, and the other a

segment q inches high. Find the ratio of the areas of the two sections made by the planes, and show that the areas are proportional to the rectangles of the parts into which the diameter is divided.

Suppose the segment p inches high to be cut off. Join the centre of the circular section with the centre of the sphere, and the centre of the section and centre of the sphere with the edge of the section.

Let C be the centre of the sphere, A the centre of the section, B the point on the edge at which the lines to A and C are drawn.

Then ABC is a right-angled triangle.

Let a = radius of the sphere.

Then $BC = a$, $CA = a - p$. Let $x = AB$.

$$\text{But } BC^2 = AC^2 + AB^2;$$
$$\therefore a^2 = (a-p)^2 + x^2;$$
$$\therefore a^2 = a^2 - 2ap + p^2 + x^2;$$
$$\therefore x^2 = 2ap - p^2 = p(2a-p);$$

$$\therefore \text{area of this section} = 4p(2a-p)\frac{\pi}{4} = p(2a-p)\,\pi.$$

Similarly, the area of the section made by the second plane will be $q(2a-q)\,\pi$;

$$\therefore \text{ratio of areas is } p(2a-p)\,\pi : q(2a-q)\,\pi$$
$$:: p(2a-p) : q(2a-q)$$
$$:: \frac{p}{q} : \frac{2a-q}{2a-p}.$$

Let P, Q be the parts of the diameter left after cutting off the segments.

Then $P = 2a - p$, $Q = 2a - q$;

$$\therefore \text{areas are as } \frac{p}{q} : \frac{Q}{P} :: pP : qQ.$$

Ex. 16. A hemisphere and a cone stand on equal bases, and have equal altitudes. A plane parallel to their bases cuts off a segment from each of them, whose altitude is p inches. Find the ratio of the areas of the sections made by the plane, and show that no plane can cut so as to make the sections equal, unless it coincides with the bases.

Let a be the radius of the hemisphere.

Each section will be a circle, and by the former example the area of the section of the hemisphere will be

$$4p(2a-p)\frac{\pi}{4}=p(2a-p)\pi.$$

Let DCF be the section of the cone.

Then $AC=p$, $AB=a$, $EB=a$.

Let $DC=x$.

Then $AC:AB::DC:EB$,

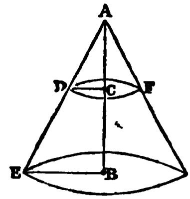

Fig. 11.

$$p:a::x:a;$$

$$\therefore p=x;$$

$$\therefore 2p=2x, \text{ the diameter of DCF};$$

$$\therefore \text{area}=(2p)^2\frac{\pi}{4}=4p^2\frac{\pi}{4}=\pi . p^2;$$

$$\therefore \text{ratio of areas is } p(2a-p)\pi : \pi p^2 :: 2a-p : p.$$

Now if the sections are to be equal, $2a-p=p$;

$$\therefore 2a=2p; \therefore a=p$$

—that is, the bases have equal areas, but no two sections made by the same plane can be equal.

Ex. 17. A ball of wood, whose specific gravity is 1·2, has a part of the inside cut out, so that a cone of metal may be put in; the cone exactly fits the hole made for it, and its height and diameter are each equal to the radius of the ball. Find the specific gravity of the metal, if the weight of the ball is exactly double what it was before.

Let a = radius of the ball.

Then its solidity will be

$$(2a)^3\times\frac{\pi}{6}=\frac{8a^3\times\pi}{6}=\frac{4a^3\pi}{3}.$$

$$\text{Solidity of cone}=\tfrac{1}{3}\left(a^2\times\frac{\pi}{4}\times a\right)=\frac{\pi a^3}{12};$$

$\therefore$ part of ball cut away to fit cone in

$$=\frac{\pi a^3}{12}\div\frac{4a^3\pi}{3}=\frac{\pi a^3\times 3}{12\times 4a^3\pi}=\tfrac{1}{16}\text{th of whole ball.}$$

$$\text{But weight of ball}=\frac{4a^3\pi}{3}\times 1.2.$$

Weight of wood left after cutting

$$=\frac{15}{16}\times\frac{4a^3\pi}{3}\times 1.2.$$

Let x = specific gravity of metal.

Then weight of cone $=\frac{\pi a^3 . x}{12}$;

$$\therefore \frac{\pi a^3 x}{12}+\frac{15}{16}\times\frac{4a^3\pi}{3}\times 1.2 = \text{twice weight of ball}$$

$$=\frac{8a^3\pi}{3}\times 1.2;$$

$$\therefore \frac{x}{12}+\frac{15\times 4\times 1.2}{16\times 3}=\frac{8\times 1.2}{3};$$

$$\therefore \frac{x}{12}=1.2\left\{\frac{8}{3}-\frac{15\times 4}{16\times 3}\right\}$$

$$=\frac{1.2}{3}(8\times 16-15\times 4)$$

$$=\frac{1.2}{3}\left(\frac{128-60}{16}\right)=\frac{1.2}{3}\times\frac{68}{16};$$

$$\therefore x=\frac{1.2\times 68}{4}=1.2\times 17$$

= 17 times specific gravity of ball = 20·4.

Ex. 18. OA, OB, OC are three adjacent edges of a cube. Given OA = OB = OC = a, find the solid content of the pyramid OABC and the area of the triangle ABC. (1st B.A. 1872.)

First find the area of the triangle ABC.

$$AB^2 = OB^2 + OA^2 = 2a^2;$$

$$\therefore AB = a\sqrt{2}.$$

And ABC is evidently an equilateral triangle, of which CD is the altitude;

$$\therefore CD = AB\frac{\sqrt{3}}{2}$$

$$=\frac{a\sqrt{2}.\sqrt{3}}{2}=\frac{a\sqrt{6}}{2};$$

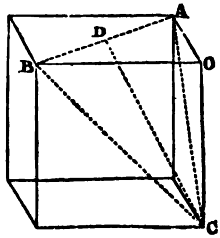

Fig. 12.

$$\therefore \text{ area} = \frac{\text{AB} \cdot \text{CD}}{2} = \frac{a\sqrt{2} \cdot a\sqrt{6}}{2 \cdot 2} = \frac{2a^2\sqrt{3}}{4} = \frac{a^2\sqrt{3}}{2};$$

$\therefore$ the base of the pyramid is an equilateral triangle;

$$\therefore \text{ its solidity} = \frac{a^2\sqrt{3}}{2} \times x \times \tfrac{1}{3} = \frac{a^2 \cdot x\sqrt{3}}{2 \times 3},$$

where x is the altitude.

We have now to find x.

Let ABC be the equilateral triangle which forms the base.

We find first the point at which the lines perpendicular to the middle points of the base cross each other.

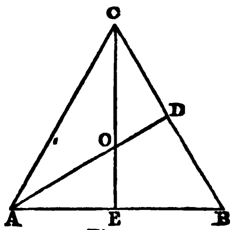

Fig. 13.

Let P be the point. This point will have O in the preceding figure vertically above it.

Since ABC is equilateral, AD, CE bisect the angles CAB, ACB, and are perpendicular to the base; $\therefore$ ACE is a right-angled triangle; so also is ACD; and if b be the side AB, we have

$$\text{AE} : \text{EP} :: \text{CE} : \text{AE, or } \frac{b}{2} : \text{EP} :: \frac{b\sqrt{3}}{2} : \frac{b}{2};$$

$$\therefore \text{ EP} = \frac{b^2}{4} \div \frac{b\sqrt{3}}{2} = \frac{b^2}{4} \times \frac{2}{b\sqrt{3}} = \frac{b}{2\sqrt{3}}.$$

But $b = a\sqrt{2}$;

$$\therefore \text{ EP} = \frac{a\sqrt{2}}{2\sqrt{3}} = \frac{a}{\sqrt{6}};$$

$$\therefore \text{ CP} = \frac{b\sqrt{3}}{2} - \frac{b}{2\sqrt{3}} = \frac{b}{2}\left\{\sqrt{3} - \frac{1}{\sqrt{3}}\right\}$$

$$= \frac{b}{2}\left\{\frac{3-1}{\sqrt{3}}\right\} = \frac{2}{\sqrt{3}}\frac{b}{2} = \frac{b}{\sqrt{3}};$$

$$\therefore \text{ CP} = \frac{a\sqrt{2}}{\sqrt{3}}.$$

Let OABC be the pyramid, OP the altitude.

Then $EP = \frac{a}{\sqrt{6}}$,

$CP = \frac{a\sqrt{2}}{\sqrt{3}}$,

$OC = a$.

But $OP^2 + PC^2 = OC^2$;

$\therefore\ x^2 + \left(\frac{a\sqrt{2}}{\sqrt{3}}\right)^2 = a^2$;

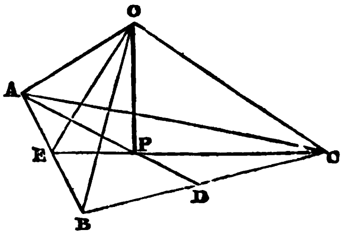

Fig. 14.

$$\therefore\ x^2 + \frac{2a^2}{3} = a^2;$$

$$\therefore\ x^2 = \frac{a^2}{3};$$

$$\therefore\ x = \frac{a}{\sqrt{3}};$$

$$\therefore \text{ solidity of pyramid } \frac{a^2\sqrt{3}}{2} \times \frac{a}{\sqrt{3}} \times \frac{1}{3} = \frac{a^3}{6}.$$

It is therefore one-sixth of the solidity of the whole cube.

A knowledge of trigonometry being essential in studying co-ordinate geometry, we shall defer this subject for the present, and begin plane trigonometry in the next chapter.

IV. TRIGONOMETRY.

The study of trigonometry must precede that of co-ordinate geometry. There are several text-books well suited to the syllabus. The following are amongst the best:—

Galbraith and Haughton's, published by Cassell, Petter, & Galpin; Todhunter's, published by Macmillan; Snowball's, published by Macmillan; Garnett's, published by Stewart & Co.

If the first of these text-books is used, the student cannot do better than work through the book, chapter after chapter, to the end of the Solution of Triangles.

If Todhunter's is adopted, chapters i. to vi., and viii. to xvi., will be sufficient. If Snowball's is selected, Articles 1 to 58, 69 to 96, and Appendices i. to iv. must be studied.

If the latter book is used, pages 30 to 127 will be found to contain all the subjects included in the syllabus.

Plane trigonometry, as the name implies, investigates the relations of the angles and sides of plane figures. It is chiefly concerned with the relations of the sides and angles of plane triangles.

Spherical trigonometry investigates the relations of solid angles and figures described on the surface of a sphere.

Euclid takes two right angles as the superior limit of magnitude of an angle. In trigonometry an angle may have any magnitude whatever; we may have an angle of 50 degrees or 500 degrees.

The symbol π is used to represent two right angles. Thus π represents an angle of 180 degrees, $\frac{\pi}{2}$ an angle of 90 degrees, $\frac{\pi}{3}$ an angle of 60 degrees, $2\pi + 20$ an angle of 380 degrees, and so on.

There are three modes of expressing the magnitude of an angle.

(1) The English method, by degrees, minutes, and seconds. Degrees are indicated by a small circle on the right of a number; minutes, by an accent on the right, drawn from right to left; seconds, by two accents in the same position, and in the same direction.

Thus $29^\circ\ 30'\ 45''$ means 29 degrees, 30 minutes, and 45 seconds.

By this method a right angle is divided into $60 \times 60 \times 90$, or 324,000 parts called seconds.

(2) The Continental method, by grades, minutes, and seconds. Grades are indicated by the letter *g* on the right of figures expressing the magnitude of the angle; minutes and seconds by accents, as in the English method, except that the accents are drawn from left to right. Thus $29^g\ 30^{`}\ 45^{``}$ would be read 29 grades, 30 minutes, 45 seconds.

100 minutes = 1 grade,
100 seconds = 1 minute.

hus the Continental method is decimal, and on this :count it has certain advantages.

Thus $29^g\ 30^`\ 45^{``}$ may be expressed at once as grades ıd decimals of a grade, thus:

$$29^g\ 30^`\ 45^{``} = 29{\cdot}3045 \text{ grades},$$
$$29^g\ 3^`\ 5^{``} = 29{\cdot}0305 \text{ grades}.$$

Degrees are reduced to grades by multiplying by $\frac{10}{9}$, ır 100 grades = 90 degrees.

Similarly, grades are reduced to degrees by multiplying y $\frac{9}{10}$.

The degree second is the $\frac{1}{90 \times 60 \times 60}$ of a right angle.

The grade second is $\frac{1}{100 \times 100 \times 100}$ of a right angle;

$$\therefore 100 \times 100 \times 100 \text{ grade seconds}$$
$$= 90 \times 60 \times 60 \text{ degree seconds};$$
$$\therefore 250 \text{ grade seconds} = 81 \text{ degree seconds};$$
$$\therefore 1 \text{ grade second} = \frac{81}{250} \text{ degree seconds},$$
$$1 \text{ degree second} = \frac{250}{81} \text{ grade seconds}.$$

Hence we may reduce grade seconds to degree seconds ›y multiplying by $\frac{81}{250}$ or $\cdot 324$, and degree seconds to grade seconds by multiplying by $\frac{250}{81}$.

(3) Circular measure. Since the ratio of the radius :o the circumference is constant whatever be the magni- :ude of a circle, it is clear that the angle at the centre of a circle, which subtends an arc equal in length to the :adius, will be constant. This angle is called the unit of circular measure. Its value is

$$57{\cdot}2957^\circ, \text{ or } 57^\circ\ 17'\ 44{\cdot}7'';$$
$$63{\cdot}66197^g, \text{ or } 63^g\ 66^`\ 19{\cdot}7^{``}.$$

Or, taking $3{\cdot}1416$ as the ratio of the circumference to the diameter of a circle, the unit is

$$\frac{180^\circ}{3{\cdot}1416}, \text{ or } \frac{200^g}{3{\cdot}1416}.$$

H

Hence, in three units of circular measure, there would be $\frac{3 \times 180}{3 \cdot 1416}$ degrees, or $\frac{3 \times 200}{3 \cdot 1416}$ grades, and in 90° there would be $90 \div \frac{180}{3 \cdot 1416}$ units—that is, $\frac{90 \times 3 \cdot 1416}{180} = \frac{3 \cdot 1416}{2}$ units $= 1 \cdot 5708$ units, and so on.

The next step is to learn the ratios represented by the functions of an angle, as the sine, cosine, tangent, cotangent, secant, cosecant, versine. These terms represent ratios between certain lines; or, if the angle of which these functions are used be placed at the centre of a circle, and the radius of the circle be taken as unity, the ratios may be represented graphically by certain lines, thus:—

Let ABCD be a circle, of which O is the centre.

Draw two diameters, AOB, DOC, perpendicular to each other. Draw OGH, making an angle α with the radius OD, and an angle β with the radius AO.

Then the two angles $\alpha + \beta$ = one right angle.

From G, where the line OGH cuts the circumference, draw GF perpendicular to OD. At D draw DE perpendicular to OD, meeting OGH in E. At A draw AH perpendicular to AO, meeting OGH in H.

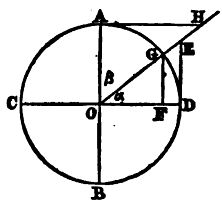

Fig. 15.

This figure will show all the functions of the angles α and β.

$$\text{Sine } \alpha \text{ written } \sin \alpha = \frac{GF}{OG}.$$

$$\text{Cosine } \alpha \text{ written } \cos \alpha = \frac{OF}{OG}.$$

$$\text{Tan } \alpha = \frac{DE}{OD}.$$

$$\text{Cotan } \alpha = \frac{AH}{AO}.$$

$$\text{Secant } \alpha \text{ or } \sec \alpha = \frac{EO}{OD}.$$

$$\text{Cosecant } \alpha \text{ or } \operatorname{cosec} \alpha = \frac{OH}{AO}.$$

$$\text{Versine } a \text{ or versin } a = \frac{FD}{OD} = \frac{OD - OF}{OD}$$

$$= 1 - \frac{OF}{OD} = 1 - \frac{OF}{OG} = 1 - \text{cosine } a.$$

Now, if we regard the radius as unity, the functions nay be remembered as follows:—

Sin a = GF.	Sec a = EO.
Cos a = OF.	Cosec a = OH.
Tan a = DE.	Versin a = FD.
Cotan a = AH.	

This is a convenient method of remembering them, but it must not be forgotten that the real value of these functions is the ratio between these lines and the radius of the circle.

By this figure, many other important principles of trigonometry may be proved.

(1) The sine, tangent, and secant of an angle a = cosine, cotangent, cosecant of $90 - a$.

The angle $\beta = 90 - a$.

Let a perpendicular, GK, be drawn from G on AO.

Then GK will be the sine of β or of $90 - a$.

But GK = OF; ∴ GK = cos a;

$$\therefore \sin (90 - a) = \cos a.$$

Similarly, AH is the tangent of β, or $90 - a$;

$\therefore \tan (90 - a) = \cot a$, and $\sec (90 - a) = \operatorname{cosec} a$.

(2) The variation in magnitude and sign of the sine and cosine may be traced.

Suppose the angle a to be very small. Then the point G will be near D, and F will be near D.

Now, if a increase, G and F will both recede from D.

Hence, GF will increase, and OF will decrease; ∴ as the angle increases, the sine increases, and the cosine decreases.

Suppose the angle increases to 90°. G will then coincide with A, and F with O.

Hence, at 90° the sine will be equal to the radius, and the cosine will be zero;

∴ for an angle of 90°, $\sin 90 = 1$, $\cos 90 = 0$.

Again, let the angle a exceed 90. Then G will be in the second quadrant, and as the angle increases towards 180, the sine will decrease and the cosine increase beyond the centre O—that is, the cosine increases from zero in a negative direction.

At 180, G will coincide with C. Hence the cosine will be equal to the radius, but negative, and the sine will be zero.

Similarly, at 270°, the cosine will again be zero, and the sine -1; and at 360° the cosine will be $+1$, and the sine 0.

(3) This figure is also a convenient one for finding the relations between the ratios.

The triangles OGF, OED, are similar;

$$\therefore \frac{GF}{OF}=\frac{ED}{OD};$$

$$\therefore \frac{GF}{OG}\div\frac{OF}{OG}=\frac{ED}{OD};$$

$$\therefore \sin \alpha \div \cos \alpha = \tan \alpha,$$

$$\text{or } \tan \alpha = \frac{\sin \alpha}{\cos \alpha}. \qquad (1)$$

In the same way, $\dfrac{OE}{OD}=\dfrac{OG}{OF}$;

$$\therefore \sec \alpha = \frac{1}{\cos \alpha}, \qquad (2)$$

$$\text{and } OG^2 = GF^2 + OF^2;$$

$$\therefore \sin^2 \alpha + \cos^2 \alpha = 1. \qquad (3)$$

$$AH^2 + AO^2 = OH^2;$$

$$\therefore \cot^2 \alpha + 1 = \operatorname{cosec}^2 \alpha. \qquad (4)$$

$$OE^2 = ED^2 + OD^2;$$

$$\therefore \sec^2 \alpha = \tan^2 \alpha + 1. \qquad (5)$$

$$\frac{AH}{AO}=\frac{OF}{GF};$$

$$\therefore \cot \alpha = \frac{\cos \alpha}{\sin \alpha} = \frac{1}{\tan \alpha}. \qquad (6)$$

$$\text{Similarly, } \sin \alpha = \frac{1}{\operatorname{cosec} \alpha}; \qquad (7)$$

and it has been shown that

$$\operatorname{versin} \alpha = 1 - \cos \alpha. \qquad (8)$$

These formulæ must be carefully committed to memory. The variations in magnitude and sign of all the other functions of an angle may now be studied.

Thus, the sine varies in the first quadrant from 0 to 1, and the cosine from 1 to 0.

But $\tan \alpha = \dfrac{\sin \alpha}{\cos \alpha}$;

$\therefore$ in the first quadrant the tangent varies from $\frac{0}{1}$ to $\frac{1}{0}$—that is, it is positive, and varies from 0 to infinity, for $\frac{0}{1}=0$, and $\frac{1}{0}=$ infinity.

In the second quadrant sine varies from 1 to 0, and cosine from 0 to -1;

$$\therefore \text{ tangent varies from } \frac{1}{0} \text{ to } \frac{0}{-1},$$

that is, from infinity to -0.

In the third quadrant sine varies from 0 to -1, cosine from -1 to 0;

$\therefore$ tangent varies from $\frac{0}{-1}$ to $\frac{-1}{0}$, or from -0 to $-$infinity.

In the fourth quadrant sine varies from -1 to 0, cosine from 0 to 1;

$\therefore$ tangent varies from $\frac{-1}{0}$ to $\frac{0}{1}$, or from $-$infinity to 0.

By the use of these formulæ the magnitude and sign of all the other functions may be traced in a similar manner.

We may also by substitution express the value of any one of the functions in terms of any other.

Ex. 1. Find tan A in terms of sin A, and also in terms of cos A.

$$\text{Tan } A=\frac{\sin A}{\cos A}.$$

$$\text{But } \sin^2 A+\cos^2 A=1;$$

$$\therefore \sin^2 A=1-\cos^2 A;$$

$$\therefore \sin A=\pm\sqrt{1-\cos^2 A},$$

$$\text{and } \cos A=\pm\sqrt{1-\sin^2 A};$$

$\therefore$ by substitution,

$$\tan A=\pm\frac{\sin A}{\sqrt{1-\sin^2 A}}$$

$$=\pm\frac{\sqrt{1-\cos^2 A}}{\cos A}.$$

We may also find the arithmetical value of any of these functions when the value of one of them is given.

Ex. 2. Find the value of sin A, cos A, cosec A, sec A, tan A, and versin A when cot $A=\frac{3}{2}$.

$$\text{Since } \cot A=\frac{\cos A}{\sin A};$$

$$\therefore \frac{\cos A}{\sin A}=\frac{3}{2};$$

$$\therefore \frac{\sqrt{1-\sin A}}{\sin A}=\frac{3}{2};$$

$$\therefore \frac{1-\sin^2 A}{\sin^2 A}=\frac{9}{4};$$

$$\therefore 4-4\sin^2 A=9\sin^2 A;$$

$$\therefore 13\sin^2 A=4;$$

$$\therefore \sin A=\pm\frac{2}{\sqrt{13}}=\frac{2\sqrt{13}}{13}.$$

$$\text{Again, } \frac{\cos A}{\sqrt{1-\cos^2 A}}=\frac{3}{2};$$

$$\therefore \frac{\cos^2 A}{1-\cos^2 A}=\frac{9}{4};$$

$$\therefore 4\cos^2 A=9-9\cos^2 A;$$

$$\therefore 13\cos^2 A=9;$$

$$\therefore \cos A=\frac{3}{\sqrt{13}}=\frac{3\sqrt{13}}{13}.$$

$$\text{Again, } \sec A=\frac{1}{\cos A}=\frac{1}{\frac{3}{\sqrt{13}}}=\frac{\sqrt{13}}{3},$$

$$\tan A=\frac{1}{\cot A}=\frac{2}{3},$$

$$\operatorname{cosec}^2 A=1+\cot^2 A$$

$$=1+\frac{9}{4}=\frac{13}{4};$$

$$\therefore \operatorname{cosec} A=\frac{1}{2}\sqrt{13}.$$

$$\text{Versin } A=1-\cos A=1-\frac{3}{\sqrt{13}}=\frac{\sqrt{13}-3}{\sqrt{13}}$$

$$=\frac{13-3\sqrt{13}}{13}.$$

The following are examples of identities proved by these formulæ:—

Ex. 3. Show that

$$\sec^2 A \operatorname{cosec}^2 A = \sec^2 A + \operatorname{cosec}^2 A.$$

By (1), $\operatorname{cosec} A = \frac{1}{\sin A}$; and by (2), $\sec A = \frac{1}{\cos A}$;

$$\therefore \sec^2 A \operatorname{cosec}^2 A = \frac{1}{\sin^2 A} \cdot \frac{1}{\cos^2 A} = \frac{1}{\sin^2 A \cos^2 A}.$$

$$\text{But } 1 = \sin^2 A + \cos^2 A;$$

$$\therefore \sec^2 A \operatorname{cosec}^2 A = \frac{\sin^2 A + \cos^2 A}{\sin^2 A \cos^2 A}$$

$$= \frac{\sin^2 A}{\sin^2 A \cos^2 A} + \frac{\cos^2 A}{\sin^2 A \cos^2 A} = \frac{1}{\cos^2 A} + \frac{1}{\sin^2 A}$$

$$= \sec^2 A + \operatorname{cosec}^2 A.$$

Ex. 4. Show that versin $\beta = \frac{\sec \beta - 1}{\sec \beta}$

$$\text{Versin } \beta = 1 - \cos \beta$$

$$= 1 - \frac{1}{\sec \beta} = \frac{\sec \beta - 1}{\sec \beta}.$$

Ex. 5. Show that $\sin A \cos A = \frac{1}{\tan A + \cot A}$.

$$\text{Sin } A \cos A = \frac{\sin A . \cos A}{1} = \frac{\sin A \cos A}{\sin^2 A + \cos^2 A}$$

$$= \frac{1}{\dfrac{\sin^2 A + \cos^2 A}{\sin A \cos A}} = \frac{1}{\dfrac{\sin A}{\cos A} + \dfrac{\cos A}{\sin A}} = \frac{1}{\tan A + \cot A}.$$

From a given equation we sometimes have to find the value of a particular function of the angle.

Ex. 6. If $\tan^2 A + 4 \sin^2 A = 6$, find $\sin A$.

Substituting for $\tan^2 A$,

$$\frac{\sin^2 A}{\cos^2 A} + 4 \sin^2 A = 6;$$

$$\therefore \frac{\sin^2 A}{1 - \sin^2 A} + 4 \sin^2 A = 6;$$

$$\therefore \sin^2 A + 4 \sin^2 A - 4 \sin^4 A = 6 - 6 \sin^2 A;$$

$$\therefore 11 \sin^2 A - 4 \sin^4 A = 6;$$

$$\therefore \sin^4 A - \frac{11 \sin^2 A}{4} = -\frac{6}{4} = -\frac{3}{2}.$$

Completing the square,

$$\sin^4 A - \frac{11 \sin^2 A}{4} + \frac{121}{64} = \frac{121}{64} - \frac{3}{2} = \frac{121-96}{64}$$
$$= \frac{25}{64}.$$

Extracting the square root,

$$\sin^2 A - \frac{11}{4} = \pm\frac{5}{4};$$
$$\therefore \sin^2 A = 2 \text{ or } \frac{3}{4};$$
$$\therefore \sin A = \pm\frac{1}{2}\sqrt{3} \text{ or} \pm \sqrt{2}.$$

The last values of sin A are inadmissible, since sin A cannot be greater than 1.

Ex. 7. If $\tan A + \sin A = m$, and $\tan A - \sin A = n$, find cos A.

$$\text{Tan } A + \sin A = m$$
$$\tan A - \sin A = n.$$

Adding, $2 \tan A = m + n$.
Subtracting, $2 \sin A = m - n$;

$$\therefore \frac{2 \tan A}{2 \sin A} = \frac{m+n}{m-n}.$$

But $\dfrac{\tan A}{\sin A} = \dfrac{\sin A}{\cos A} \div \sin A = \cos A$;

$$\therefore \cos A = \frac{m-n}{m+n}.$$

Ex. 8. If $\cos x = \dfrac{\cos A}{\sin C}$, and $\cos(90 - x) = \dfrac{\cos B}{\sin C}$, show that $\cos^2 A + \cos^2 B + \cos^2 C = 1$.

$$\text{If } \cos x = \frac{\cos A}{\sin C};$$
$$\therefore \cos^2 x = \frac{\cos^2 A}{\sin^2 C};$$
$$\therefore 1 - \sin^2 x = \frac{\cos^2 A}{\sin^2 C}.$$

But $\cos(90 - x) = \sin x$;
$\therefore$ by the second equation,

$$\sin^2 x = \frac{\cos^2 B}{\sin^2 C}.$$

Substituting in the former result,

$$1-\frac{\cos^2 B}{\sin^2 C}=\frac{\cos^2 A}{\sin^2 C}.$$

And $\sin^2 C=1-\cos^2 C$;

$$\therefore\ 1-\frac{\cos^2 B}{1-\cos^2 C}=\frac{\cos^2 A}{1-\cos^2 C};$$

$$\therefore\ 1-\cos^2 C-\cos^2 B=\cos^2 A;$$

$$\therefore\ \cos^2 A+\cos^2 B+\cos^2 C=1.$$

The values of the functions of some angles may be found by a simple method.

Let OAB be an isosceles right-angled triangle, then AOB is 45°, and BAO is 45°.

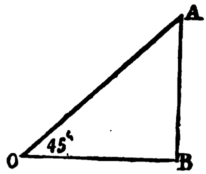

Fig. 16.

By Euclid I. 47,

$$AO^2=AB^2+BO^2;$$

$$\therefore\ AO^2=2AB^2;$$

$$\therefore\ \frac{AB}{AO}=\frac{1}{\sqrt{2}},\text{ and }\frac{OB}{AO}=\frac{1}{\sqrt{2}}.$$

But $\dfrac{AB}{AO}=\sin AOB=\sin 45$;

$$\therefore\ \sin 45=\frac{1}{\sqrt{2}}.$$

And $\cos 45=\dfrac{OB}{AO}=\dfrac{1}{\sqrt{2}}$,

$$\tan 45=\frac{\sin 45}{\cos 45}=1,$$

$$\cot 45=\frac{\cos 45}{\sin 45}=1,$$

$$\sec 45=\frac{1}{\cos 45}=\sqrt{2},$$

$$\text{cosec } 45=\frac{1}{\sin 45}=\sqrt{2},$$

$$\text{versin } 45=1-\cos 45=1-\frac{1}{\sqrt{2}}$$

$$=\frac{\sqrt{2}-1}{\sqrt{2}}=\tfrac{1}{2}(2-\sqrt{2}).$$

Again, let ABC be an equilateral triangle.

Draw AD perpendicular to BC, which will bisect the base BC.

Then $\angle$ ABD = 60, $\angle$ BAD = 30, and by Euclid I. 47,

$$AB^2 = AD^2 + BD^2.$$

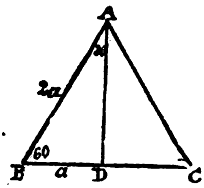

Fig. 17.

Let $AB = 2a$;

$$\therefore BD = a;$$
$$\therefore (2a)^2 = AD^2 + a^2;$$
$$\therefore 4a^2 = AD^2 + a^2;$$
$$\therefore 3a^2 = AD^2;$$
$$\therefore AD = a\sqrt{3};$$
$$\therefore \frac{AD}{AB} = \frac{a\sqrt{3}}{2a} = \frac{1}{2}\sqrt{3};$$
$$\therefore \sin 60 = \frac{1}{2}\sqrt{3}.$$

But $\sin 60 = \cos(90-60) = \cos 30$;

$$\therefore \cos 30 = \frac{1}{2}\sqrt{3},$$

$$\text{and } \frac{BD}{AB} = \frac{a}{2a} = \frac{1}{2} = \cos 60;$$

$$\therefore \cos 60 = \frac{1}{2}, \ \sin 30 = \frac{1}{2};$$

$$\therefore \tan 60 = \frac{\sin 60}{\cos 60} = \frac{\frac{1}{2}\sqrt{3}}{\frac{1}{2}} = \sqrt{3},$$

$$\tan 30 = \frac{\sin 30}{\cos 30} = \frac{\frac{1}{2}}{\frac{1}{2}\sqrt{3}} = \frac{1}{\sqrt{3}}$$

The values of the sines and cosines of 30°, 45°, 60°, should be carefully remembered.

	Sin.	*Cos.*
30,	$\frac{1}{2}$.	$\frac{1}{2}\sqrt{3}$.
45,	$\frac{1}{\sqrt{2}}$.	$\frac{1}{\sqrt{2}}$.
60,	$\frac{1}{2}\sqrt{3}$.	$\frac{1}{2}$.

Sin A = sin (180 − A), cos A = −cos (180 − A).

Combining these with the values of sin 30, etc., we get

$$\sin (180 - 30) = \sin 30 ;$$

$$\therefore \sin 150 = \sin 30 = \frac{1}{2},$$

$$\cos 150 = -\cos 30 = -\frac{1}{2}\sqrt{3},$$

$$\sin (180 - 45) = \sin 135 = \sin 45 = \frac{1}{\sqrt{2}},$$

$$\cos 135 = -\cos 44 = \frac{1}{\sqrt{2}},$$

$$\sin (180 - 60) = \sin 120 = \sin 60 = \frac{1}{2}\sqrt{3},$$

$$\cos 120 = -\cos 60 = -\frac{1}{2}.$$

We have thus obtained the values of the sine and cosine of 30°, 45°, 60°, 120°, 135°, 150°.

We may observe that the angle A in Ex. 6 was 120°.

Ex. 9. Solve the equation,

$$\sin^2 \theta + \cos^2 (90 - \theta) = 1.$$

$$\cos (90 - \theta) = \sin \theta ;$$

$$\therefore \sin^2 \theta + \sin^2 \theta = 1 ;$$

$$\therefore 2 \sin^2 \theta = 1,$$

$$\sin^2 \theta = \frac{1}{2} ;$$

$$\therefore \sin \theta = \pm \frac{1}{\sqrt{2}} ;$$

∴ the angle is 45° or 225°.

Ex. 10. Find the angle whose tangent is twice its sine.

Let θ be the angle.

Then $\tan \theta = 2 \sin \theta$;

$$\therefore \frac{\sin \theta}{\cos \theta} = 2 \sin \theta ;$$

$$\therefore \frac{1}{\cos \theta} = 2 ;$$

$$\therefore 2 \cos \theta = 1 ;$$

$$\therefore \cos \theta = \tfrac{1}{2} ;$$

∴ the angle = 60°.

Ex. 11. Find the angle whose sine and cosine are together equal to unity.

Let x be the angle.

Then $\sin x + \cos x = 1$;

$$\therefore \sin x + \sqrt{1 - \sin^2 x} = 1;$$
$$\therefore \sqrt{1 - \sin^2 x} = 1 - \sin x;$$
$$\therefore 1 - \sin^2 x = 1 - 2 \sin x + \sin^2 x;$$
$$\therefore 2 \sin^2 x = 2 \sin x;$$
$$\therefore \sin x = 0 \text{ or } 1;$$
$$\therefore \text{the angle is } 0 \text{ or } 90.$$

It sometimes simplifies the process if a single letter is substituted for a function.

Ex. 12. If the tangent of an angle is $\frac{m}{n}$, find the sine.

Let θ be the angle.

$$\text{Tan } \theta = \frac{m}{n};$$
$$\therefore \frac{\sin \theta}{\cos \theta} = \frac{m}{n};$$
$$\therefore \frac{\sin \theta}{\sqrt{1 - \sin^2 \theta}} = \frac{m}{n}.$$

Let $x = \sin \theta$;

$$\therefore \frac{x}{\sqrt{1 - x^2}} = \frac{m}{n};$$
$$\therefore \frac{x^2}{1 - x^2} = \frac{m^2}{n^2};$$
$$\therefore n^2x^2 = m^2 - m^2x^2;$$
$$\therefore x^2(m^2 + n^2) = m^2;$$
$$\therefore x^2 = \frac{m^2}{m^2 + n^2};$$
$$\therefore x \text{ or } \sin \theta = \frac{\pm m}{\sqrt{m + n}}.$$

Ex. 13. If $n = \tan A + \sin A$, . . . (1)
and $m = \tan A - \sin A$, find an equation connecting m and n.

By addition, $\tan A = \frac{m+n}{2}$. . . . (3)

raction, $\sin A = \dfrac{m-n}{2}$ . . . . (4)

$$\text{But } \tan A = \frac{\sin A}{\cos A} = \frac{m+n}{2};$$

$$\therefore \frac{m-n}{2\cos A} = \frac{m+n}{2};$$

$$\therefore \cos A = \frac{m-n}{m+n}.$$

$$\text{Again, } \frac{\sqrt{1-\cos^2 A}}{\cos A} = \frac{m+n}{2};$$

$$\therefore \frac{1-\cos^2 A}{\cos^2 A} = \frac{(m+n)^2}{4};$$

$$\therefore \frac{1}{\cos^2 A} - 1 = \frac{(m+n)^2}{4};$$

$$\therefore \frac{1}{\cos^2 A} = \frac{(m+n)^2+4}{4}.$$

former result gives $\dfrac{1}{\cos^2 A} = \dfrac{(m+n)^2}{(m-n)^2}$;

$$\therefore \left(\frac{m+n}{m-n}\right)^2 = \frac{(m+n)^2+4}{4};$$

$$\therefore 4(m+n)^2 = (m^2-n^2)^2 + 4(m-n)^2;$$

$$^2 + 2mn + n^2) = (m^2-n^2)^2 + 4(m^2 - 2mn + n^2);$$

$$\therefore 16mn = (m^2-n^2)^2.$$

. Having given that $\tan A + \cot A = \frac{4}{3}\sqrt{3}$, ›1,030, and $\log 3 = 477{,}121$, find A, log sin A, , log tan A, and the log of $\cot^2 A - \cos^2 A$.

$$\text{Tan } A + \cot A = \frac{4}{3}\sqrt{3};$$

$$\therefore \tan A + \frac{1}{\tan A} = \frac{4}{3}\sqrt{3};$$

$$\therefore \tan^2 A + 1 = \frac{4}{3}\sqrt{3} \cdot \tan A;$$

$$\therefore \tan^2 A - \frac{4}{3}\sqrt{3} \tan A = -1.$$

Completing square,

$$\tan^2 A - \frac{4}{3}\sqrt{3} \tan A + \frac{4}{3} = \frac{1}{3};$$

$$\therefore \tan A - \frac{2}{\sqrt{3}} = \pm \frac{1}{\sqrt{3}};$$

$$\therefore \tan A = \frac{3}{\sqrt{3}} = \sqrt{3} \text{ or } \frac{1}{\sqrt{3}};$$

$$\therefore A = 60^\circ \text{ or } 30^\circ;$$

$$\therefore \sin A = \tfrac{1}{2} \text{ or } \tfrac{1}{2}\sqrt{3};$$

$$\begin{aligned} \therefore \log \sin A &= 0 - \log 2 = -\cdot 301030 + 10 \\ &= 9\cdot 698970, \end{aligned}$$

$$\begin{aligned} \text{or } \tfrac{1}{2} \log 3 - \log 2 &= \cdot 238560 - \cdot 301030 + 10 \\ &= 10\cdot 238560 - \cdot 301030 \\ &= 9\cdot 937530, \end{aligned}$$

log cos A.

Since the angle may be 30 or 60, the cosine of one will be the sine of the other;

$$\therefore \log \cos A = 9\cdot 937530 \text{ or } 9\cdot 698970.$$

$$\begin{aligned} \text{Log} \tan A &= 0 - \tfrac{1}{2} \log 3 = 0 - \cdot 238560 + 10 \\ &= 9\cdot 761440, \\ &\text{or } \tfrac{1}{2} \log 3 + 10 = 10\cdot 238560. \end{aligned}$$

$$\begin{aligned} \text{Cot}^2 A - \cos^2 A &= \frac{\cos^2 A}{\sin^2 A} - \cos^2 A \\ &= \cos^2 A \left(\frac{1}{\sin^2 A} - 1\right) \\ &= \frac{\cos^2 A}{\sin^2 A} (1 - \sin^2 A) \\ &= \frac{\cos^2 A}{\sin^2 A} \cdot \cos^2 A = \frac{\cos^4 A}{\sin^2 A}; \end{aligned}$$

$$\begin{aligned} \therefore \log &= 4(\log \cos A - 10) - 2(\log \sin A - 10) \\ &= 4 \log \cos A - 2 \log \sin A - 20 \\ &= 4 \times 9\cdot 937530 - 2 \times 9\cdot 698970 - 20, \end{aligned}$$

$$\begin{aligned} 39\cdot 750120 - 19\cdot 397940 - 20 &= 39\cdot 750120 - 39\cdot 397940 \\ &= \cdot 352180. \end{aligned}$$

FORMULÆ INVOLVING TWO OR MORE ANGLES.

In the proofs of the formulæ representing the functions of the sum and difference of two angles, the geometrical proof must be studied, and the figures must be constructed for the cases when—

(1) A, B, and $A+B$ are all less than a right angle.

(2) A and B each less than one right angle, and $A+B$ greater than one right angle.

(3) A and B each greater than a right angle, etc.

The values $\sin(A+B)=\sin A\cos B+\cos A\sin B$,
$$\cos(A+B)=\cos A\cos B-\sin A\sin B,$$
being found geometrically, the values of $\sin(A-B)$ and $\cos(A-B)$ may be found either geometrically or by substitution in the above.

Thus, $\sin(A-B)=\sin A\cos(-B)+\cos A\sin(-B)$.

$$\text{Now } \cos(-B)=\cos B,$$
$$\sin(-B)=-\sin B;$$
$$\therefore \sin(A-B)=\sin A\cos B-\cos A\sin B.$$

Similarly, $\cos(A-B)=\cos A\cos(-B)-\sin A\sin(-B)$
$$=\cos A\cos B+\sin A\sin B.$$

Tan $(A\pm B)$ are found from $\dfrac{\sin(A\pm B)}{\cos(A\pm B)}$.

$$\operatorname{Tan}(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}.$$

$$\operatorname{Tan}(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}.$$

And since $\cot A=\dfrac{1}{\tan A}$,

$$\therefore \cot(A\pm B)\,\frac{1\mp\tan A\tan B}{\tan A\pm\tan B}.$$

From the values of the functions of 45°, 30°, and 60° on p. 122, we may find the functions of many others; for example,

$$\sin 75=\sin(45+30)=\sin 45\cos 30+\cos 45\sin 30$$
$$=\frac{1}{\sqrt{2}}\cdot\tfrac{1}{2}\sqrt{3}+\frac{1}{\sqrt{2}}\cdot\tfrac{1}{2}$$
$$=\frac{1}{2\sqrt{2}}(\sqrt{3}+1).$$

$$\operatorname{Cos} 75=\cos(45+30)=\cos 45\cos 30-\sin 45\sin 30$$
$$=\frac{1}{\sqrt{2}}\cdot\tfrac{1}{2}\sqrt{3}-\frac{1}{\sqrt{2}}\cdot\tfrac{1}{2}=\frac{1}{2\sqrt{2}}(\sqrt{3}-1).$$

$$\text{Tan } 75 = \tan(45+30) = \frac{\tan 45 + \tan 30}{1 - \tan 45 \tan 30}$$

$$= \frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}} = \frac{\sqrt{3}+1}{\sqrt{3}-1};$$

$$\therefore \text{ also } \cot 75 = \frac{\sqrt{3}-1}{\sqrt{3}+1}.$$

In the same way the functions of $15°$ and $105°$ may be found, with others.

If B becomes A in the above, we get

$$\sin 2A = \sin(A+A) = 2 \sin A \cos A.$$
$$\cos 2A = \cos^2 A - \sin^2 A.$$

And since $\sin^2 A + \cos^2 A = 1$,

$$\cos 2A = 1 - 2 \cos^2 A$$
$$= 2 \sin^2 A - 1.$$

Also,

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}, \text{ and } \cot 2A = \frac{1 - \tan^2 A}{2 \tan A}.$$

These formulæ may be used to find the values of the functions of an angle which is double a given angle, and also the values of the functions of half a given angle.

$$\text{Thus, } \sin 120 = 2 \sin 60 \cos 60$$
$$= 2 \cdot \tfrac{1}{2}\sqrt{3} \cdot \tfrac{1}{2} = \tfrac{1}{2}\sqrt{3}.$$

$$\text{Tan } 120 = \frac{2 \cdot \tan 60}{1 - \tan^2 60}$$
$$= \frac{2 \cdot \sqrt{3}}{1-3} = -\sqrt{3}.$$

If we suppose A to become $\frac{A}{2}$, the formula

$\cos 2A = 1 - 2 \cos^2 A$, becomes $\cos A = 1 - 2 \cos^2 \frac{A}{2}$;

$$\therefore \cos^2 \frac{A}{2} = \frac{1 - \cos A}{2}.$$

Thus, from $\cos 75 = \frac{1}{2\sqrt{2}}(\sqrt{3}-1)$, we get

$$\cos^2 \frac{75}{2} = \frac{1 - \frac{1}{2\sqrt{2}}(\sqrt{3}-1)}{2} = \frac{2\sqrt{2}-\sqrt{3}+1}{4\sqrt{2}};$$

$$\therefore \cos 37\tfrac{1}{2} = \sqrt{\frac{2\sqrt{2}-\sqrt{3}+1}{2\sqrt{2}}} = \sqrt{\frac{4-\sqrt{6}+\sqrt{2}}{2}}.$$

In this way the values of the functions of a large number of angles may readily be calculated. It will be a useful exercise for the student to find the values of the functions of $22\frac{1}{2}$, 15, $37\frac{1}{2}$, and $7\frac{1}{2}$ by this method.

Expressions may be deduced from the expansions of sin (A+B), cos (A+B), etc., for the functions of the sum of three or more angles; thus,

$$\begin{aligned}
&\sin (A+B+C) \\
&= \sin \{(A+B)+C\} \\
&= \sin (A+B) \cos C + \cos (A+B) \sin C \\
&= (\sin A \cos B + \cos A \sin B) \cos C + (\cos A \cos B - \sin A \sin B) \sin C \\
&= \sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C - \sin A \sin B \sin C.
\end{aligned}$$

Similarly,

$$\cos (A+B+C) = \cos A \cos B \cos C - \sin A \sin B \cos C - \sin A \sin C \cos B - \sin B \sin C \cos A;$$

and $\tan (A+B+C)$

$$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - (\tan A \tan B + \tan A \tan C + \tan B \tan C)}.$$

If, in the above, B = C = A,

we get,
$$\begin{aligned}
\sin 3A &= \sin A \cos^2 A + \sin A \cos^2 A + \sin A \cos^2 A - \sin^3 A \\
&= 3 \sin A \cos^2 A - \sin^3 A,
\end{aligned}$$

$$\cos 3A = \cos^3 A - 3 \sin^2 A \cos A,$$

$$\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A},$$

$$\cot 3A = \frac{1 - 3 \tan^2 A}{3 \tan A - \tan^3 A}.$$

As the expressions for $\sin^2 A$, etc., enable us to find the values of the functions of half an angle, so these latter expansions will give us the means of finding the values of the functions of an angle which is one-third of a given angle.

It will be good exercise for the student to find expressions for angles of 5°, $2\frac{1}{2}$°, 10° from those already given;

and by taking advantage of the constants given on p. 175, we may find the decimals which represent the natural sine, natural cosine, tangent, etc.

It will also be a useful exercise to find expressions for sec 2A, cosec 2A, sec 3A, cosec 3A, etc., and to express sin 2A, cos 2A, tan 2A, etc., in terms of sec A, cot A, etc.

For example, it will be found that

$$\sin 2A = \frac{2 \tan A}{1 + \tan^2 A} = \frac{2\sqrt{\sec^2 A - 1}}{\sec^2 A},$$

$$\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A} = \frac{2 - \sec^2 A}{\sec^2 A} = \frac{\cot^2 A - 1}{\cot^2 A + 1},$$

From the expressions for cos 2A and sin 2A, it may be proved that

$$\cos A + \sin A = \pm \sqrt{1 + \sin 2A},$$

$$\cos A - \sin A = \pm \sqrt{1 - \sin 2A},$$

$$\frac{\tan A + \tan B}{\tan A - \tan B} = \frac{\sin (A+B)}{\sin (A-B)}.$$

The following formulæ are probably the most useful of all; they are frequently used in the higher branches of mathematics, and they should not only be carefully remembered, but the proofs should be thoroughly mastered:—

$$\text{Sin } (A+B) + \sin (A-B) = 2 \sin A \cos B.$$
$$\text{Sin } (A+B) - \sin (A-B) = 2 \cos A \sin B.$$
$$\text{Cos } (A+B) + \cos (A-B) = 2 \cos A \cos B.$$
$$\text{Cos } (A+B) - \cos (A-B) = 2 \sin A \sin B.$$
$$\text{Sin } A + \sin B = 2 \sin \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}(A-B).$$
$$\text{Sin } A - \sin B = 2 \cos \tfrac{1}{2}(A+B) \sin \tfrac{1}{2}(A-B).$$
$$\text{Cos } A + \cos B = 2 \cos \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}(A-B).$$
$$\text{Cos } A - \cos B = 2 \sin \tfrac{1}{2}(A+B) \sin \tfrac{1}{2}(A-B).$$

From these formulæ a further number of values of functions may be found; for example,

$$\sin 36 = \cos (90 - 36) = \cos 54.$$

Let $A = 18°$;

$$\therefore \sin 2A = \cos 3A;$$

$$\begin{aligned} \therefore 2 \sin A \cos A &= \cos^3 A - 3 \sin^2 A \cos A \\ &= \cos^3 A - 3(1 - \cos^2 A) \cos A \\ &= 4 \cos^3 A - 3 \cos A; \end{aligned}$$

$$\begin{aligned} \therefore 2 \sin A &= 4 \cos^2 A - 3 \\ &= 4(1 - \sin^2 A) - 3 \\ &= 1 - 4 \sin^2 A; \end{aligned}$$

$$\therefore 4 \sin^2 A + 2 \sin A = 1;$$
$$\therefore \sin^2 A + \tfrac{1}{2} \sin A = \tfrac{1}{4};$$
$$\therefore \sin^2 A + \tfrac{1}{2} \sin A + \tfrac{1}{16} = \tfrac{5}{16};$$
$$\therefore \sin A + \tfrac{1}{4} = \pm \tfrac{1}{4}\sqrt{5};$$
$$\therefore \sin A = \frac{\pm\sqrt{5} - 1}{4};$$
$$\therefore \sin A = \frac{-1 \pm \sqrt{5}}{4}.$$

From this and the preceding results a further addition may be made to the number of values of functions; for example, we may easily find those of 36°, 72°, 102°, 117°, etc.

Since $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$,

and $\tan A - B = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$;

if $B = 45°$,

$$\tan(A+45) = \frac{\tan A + 1}{1 - \tan A} = \frac{1 + \tan A}{1 - \tan A},$$

$$\tan(A-45) = \frac{\tan A - 1}{\tan A + 1};$$

and by addition,

$$\tan(A+45) + \tan(A-45) = 2 \tan 2A.$$

The following examples illustrate the use of a few of the preceding formulæ:—

Ex. 1. Determine the value of A when

$$\tan 2A = 3 \tan A.$$

$$\text{Tan } 2A = \frac{2 \tan A}{1 - \tan^2 A};$$

$$\therefore \text{by substitution, } \frac{2 \tan A}{1 - \tan^2 A} = 3 \tan A;$$

$$\therefore \frac{2}{1 - \tan^2 A} = 3;$$
$$\therefore 2 = 3 - 3 \tan^2 A;$$
$$\therefore 3 \tan^2 A = 1;$$
$$\therefore \tan A = \pm \frac{1}{\sqrt{3}};$$
$$\therefore A = 30°.$$

Ex. 2. Find A when

$$\cos 2A = \tan 2A.$$

$$\text{Tan } 2A = \frac{\sin 2A}{\cos 2A};$$

$$\therefore \cos 2A = \frac{\sin 2A}{\cos 2A};$$

$$\therefore \cos^2 2A = \sin 2A;$$

$$\therefore 1 - \cos^2 2A = 1 - \sin 2A;$$

$$\therefore \sin^2 2A = 1 - \sin 2A;$$

$$\therefore \sin^2 2A + \sin 2A = 1;$$

$$\therefore \sin^2 2A + \sin 2A + \tfrac{1}{4} = \tfrac{5}{4};$$

$$\therefore \sin 2A + \tfrac{1}{2} = \pm \tfrac{1}{2}\sqrt{5};$$

$$\therefore \sin 2A = \tfrac{1}{2}(\pm\sqrt{5} - 1) = \cdot 6180;$$

$$\therefore 2A = 38^\circ\ 10'.$$

Examples are sometimes given in which some terms appear to require expanding when such is not the case.

Ex. 3. Find x and y when

$$2 \sin (x-y) = 1, \text{ and } \sin (x-y) = \cos (x+y).$$

No expansion is necessary in this case, for if

$$2 \sin (x-y) = 1,$$

$$\sin (x-y) = \tfrac{1}{2};\ \therefore x-y = 30^\circ.$$

Again, if $\sin (x-y) = \cos (x+y)$;

$$\therefore \cos (x+y) = \tfrac{1}{2};\ \therefore x+y = 60^\circ.$$

By addition, $2x = 90^\circ$; $\therefore x = 45^\circ$.

By subtraction, $2y = 30^\circ$; $\therefore y = 15^\circ$.

If the expressions had been expanded, we should have arrived at the same results, but by a longer process; thus,

$$2 \sin (x-y) = 1;\ \therefore \sin (x-y) = \tfrac{1}{2};$$

$$\therefore \sin x \cos y - \cos x \sin y = \tfrac{1}{2}.$$

From the second equation,

$$\tfrac{1}{2} = \cos (x+y) = \cos x \cos y - \sin x \sin y;$$

$$\therefore \sin x \cos y - \cos x \sin y = \cos x \cos y - \sin x \sin y;$$

$$\therefore \sin x (\cos y + \sin y) = \cos x (\cos y + \sin y);$$

$$\therefore \sin x = \cos x;$$

$$\therefore x = 45^\circ.$$

$$\text{And } \sin (x-y) = \tfrac{1}{2};\ \therefore x-y = 30^\circ;$$

$$\therefore 45^\circ - y = 30^\circ;\ \therefore y = 15^\circ.$$

Ex. 4. Find A, B, and C when $\cos (A+B-C) = \frac{1}{2}$.

$$\text{Cos } (A-B+C) = \frac{\sqrt{3}}{2}, \text{ and } \cos (A+B) = \sin C.$$

If $\cos(A+B-C)=\frac{1}{2}$, $A+B-C=60°$.

If $\cos(A-B+C)=\frac{\sqrt{3}}{2}$, $A-B+C=30°$.

$$\text{Adding, } 2A=90°;\ \therefore A=45°;$$
$$\therefore B-C=15°.$$
Again, since $\cos(A+B)=\sin C$,
$$A+B=90°-C;$$
$$\therefore A+B+C=90;$$
$$\therefore B+C=90°-A=90°-45°=45°,$$
$$\text{and } B-C=15°;$$
$$\therefore 2B=60°;\ \therefore B=30°;$$
$$\therefore 2C=30°;\ \therefore C=15°.$$

By expanding, we sometimes get a more complete answer, as in the following case:—

Ex. 5. Find x if $\sin(a-x)=\cos(a+x)$.
Expanding,
$$\sin a \cos x-\cos a \sin x=\cos a \cos x-\sin a \sin x;$$
$$\therefore \cos x(\sin a-\cos a)=-\sin x(\sin a-\cos a);$$
$$\therefore \cos x=-\sin x,$$
$$\sqrt{1-\sin^2 x}=\sqrt{-\sin x},$$
$$1-\sin^2 x=\sin^2 x,$$
$$2\sin^2 x=1,$$
$$\sin x=\pm\frac{1}{\sqrt{2}};\ \therefore x=45° \text{ or } 135°.$$

Ex. 6. Find x when
$$\sin(x+a)+\cos(x+a)=\sin(x-a)+\cos(x-a).$$
This may be arranged
$$\sin(x+a)-\sin(x-a)=\cos(x-a)-\cos(x+a).$$
Now, by formula No. (see p.),
$$\sin(x+a)-\sin(x-a)=2\cos x \sin a.$$
And by formula No. ,
$$\cos(x-a)-\cos(x+a)=-2\sin x \sin a;$$
$$\therefore 2\cos x \sin a=-2\sin x \sin a;$$
$$\therefore \cos x=-\sin x,$$
from which $x=45°$ or $135°$, as in the preceding examples.

Ex. 7. If $\tan A=\frac{p}{q}$, and $\tan B=\frac{m}{n}$, find the value of $\sin(A-B)$, and $\cos(A-B)$.

$$\operatorname{Tan} A=\frac{\sin A}{\cos A}=\frac{p}{q};\ \therefore \frac{\sin^2 A}{\cos^2 A}=\frac{p^2}{q^2};$$

$$\therefore \frac{\sin^2 A + \cos^2 A}{\cos^2 A} = \frac{p^2+q^2}{q^2};$$

$$\therefore \frac{1}{\cos^2 A} = \frac{p^2+q^2}{q^2}; \therefore \frac{1}{\cos A} = \frac{\sqrt{p^2+q^2}}{q};$$

$$\therefore \cos A = \pm \frac{p}{\sqrt{p^2+q^2}}.$$

$$\text{Similarly, } \cos B = \pm \frac{m}{\sqrt{m^2+n^2}}.$$

$$\text{Also } \sin A = \frac{q}{p} \cos A = \pm \frac{q}{\sqrt{p^2+q^2}}.$$

$$\text{Similarly, } \sin B = \frac{n}{m} \cos B = \pm \frac{n}{\sqrt{m^2+n^2}}.$$

But $\sin(A-B) = \sin A . \cos B - \cos A \sin B$

$$= \frac{q}{\sqrt{p^2+q^2}} \cdot \frac{m}{\sqrt{m^2+n^2}} - \frac{p}{\sqrt{p^2+q^2}} \cdot \frac{n}{\sqrt{m^2+n^2}}$$

$$= \frac{mq-np}{\sqrt{(p^2+q^2)(m^2+n^2)}}.$$

$$\text{Cos } (A-B) = \frac{pm+qn}{\sqrt{(p^2+q^2)(m^2+n^2)}}.$$

Ex. 8. If the reciprocals of the cosines of $(\phi - a)$, ϕ, and $(\phi + a)$ be in arithmetic progression, express $\cos \phi$ in terms of $\cos a$.

$$\text{By question, } \frac{1}{\cos(\phi+a)} + \frac{1}{\cos(\phi-a)} = \frac{2}{\cos\phi};$$

$$\therefore \frac{\cos(\phi-a)+\cos(\phi+a)}{\cos(\phi-a)\cos(\phi+a)} = \frac{2}{\cos\phi};$$

$$\therefore \frac{2\cos\phi\cos a}{\cos(\phi-a)\cos(\phi+a)} = \frac{2}{\cos\phi};$$

$$\therefore \cos^2\phi \cos a = \cos(\phi-a)\cos(\phi+a)$$
$$= \cos^2\phi\cos^2 a - \sin^2\phi\sin^2 a;$$

$$\therefore \cos^2\phi\cos a(1-\cos a) = -\sin^2\phi\sin^2 a$$
$$= -(1-\cos^2\phi)(1-\cos^2 a);$$

$$\therefore \cos^2\phi\cos a = -(1-\cos^2\phi)(1+\cos a)$$
$$= -1+\cos^2\phi - \cos a + \cos^2\phi\cos a.$$

$$0 = \cos^2\phi - 1 - \cos a;$$

$$\therefore \cos^2\phi = 1 + \cos a.$$

Ex. 9. Having given tan $45° = 1$, tan $60° = \sqrt{3}$, and log $3732 = 3\cdot57194$, find the natural sine of $75°$ and log tan $75°$.

Tan $75° = -\tan 105°$

$$= -\tan(45+60) = \frac{-(1+\sqrt{3})}{1-\sqrt{3}} = -\frac{(1+\sqrt{3})^2}{1-3}$$

$$= \frac{1+2\sqrt{3}+3}{2} = 2+\sqrt{3}$$

$$= 2+1\cdot732 = 3\cdot1732;$$

$$\therefore \log \tan 75° = 10 + \log 3\cdot1732$$

$$= 10 + \cdot57194 = 10\cdot57194.$$

Now tan $75 = 2+\sqrt{3}$;

$$\therefore \frac{\sin 75}{\cos 75} = 2+\sqrt{3};$$

$$\therefore \frac{\sin 75}{\sqrt{1-\sin^2 75}} = 2+\sqrt{3};$$

$$\therefore \frac{\sin^2 75}{1-\sin^2 75} = 4+3+4\sqrt{3} = 7+4\sqrt{3};$$

$$\therefore \sin^2 75 = 7+4\sqrt{3} - (7+4\sqrt{3}) \sin^2 75;$$

$$\therefore (8+4\sqrt{3}) \sin^2 75 = 7+4\sqrt{3};$$

$$\therefore \sin^2 75 = \frac{7+4\sqrt{3}}{8+4\sqrt{3}};$$

$$\therefore \sin 75 = \sqrt{\frac{7+4\sqrt{3}}{8+4\sqrt{3}}}$$

$$= \frac{\sqrt{(7+4\sqrt{3})(2-\sqrt{3})}}{2}$$

$$= \frac{2+\sqrt{3}}{2} \cdot \sqrt{2-\sqrt{3}},$$

which being reduced to a decimal $= \cdot965926$, the natural sine of $75°$.

The method of calculating the functions of all angles between $1'$ and $90°$ is very clearly explained in Professor Elliott's *Logarithms and Trigonometry*, pp. 190–192, which will well repay perusal.

Before commencing the study of the formulæ relating to the solution of triangles, the student must master the proposition relating to the sines and sides of triangles—viz., the sines of the angles of a triangle are proportional to the sides respectively opposite to them.

Thus if A, B, C represent the angles of a triangle, and a, b, c the sides respectively opposite to A, B, C;

$$\therefore \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

This and $A+B+C=180^\circ$ are among the chief formulæ used in the solution of triangles.

We have six things altogether in a triangle—A, B, C, a, b, c. Any three being given (three angles excepted), the remaining three can be found.

If, however, we have two sides, and an angle opposite them, a difficulty arises which we shall notice shortly.

Next we have the formula giving the value of the cosine of an angle of a triangle in terms of its sides. The proofs of these formulæ are derived from Euclid II. 12, 13:—

$$\text{Cos } A = \frac{b^2+c^2-a^2}{2bc}.$$

$$\text{Cos } B = \frac{c^2+a^2-b^2}{2ac}.$$

$$\text{Cos } C = \frac{a^2+b^2-c^2}{2ab}.$$

It will only be necessary to remember one of these formulæ, and to notice how the others are derived from it. Thus for cos C, c^2 is subtracted from the sum of the other two squares in the numerator, and twice the product of the other two quantities, a, b, occurs in the denominator.

The following formulæ are also sometimes used in the solution of triangles, and proofs of them must be studied:—

$$\text{Cos } \tfrac{1}{2}A = \sqrt{\frac{S(S-a)}{bc}}.$$

On the same plan as above the functions of the other angles may be written down; thus:

$$\text{Cos } \tfrac{1}{2}B = \sqrt{\frac{S(S-b)}{ac}}, \text{ and so on.}$$

$$\text{Again, } \sin \tfrac{1}{2}A = \sqrt{\frac{(S-b)\,(S-c)}{bc}};$$

and from this,

$$\sin \tfrac{1}{2}B = \sqrt{\frac{(S-a)\,(S-c)}{ac}}, \text{ etc.}$$

By substitution, from the values of cos $\frac{1}{2}$A and sin $\frac{1}{2}$A we get

$$\tan \tfrac{1}{2}A = \sqrt{\frac{(S-b)(S-c)}{S(S-a)}},$$

and in the same way as before we may alter the formula to get tan $\frac{1}{2}$B and tan $\frac{1}{2}$C.

By similar substitutions we may get

$$\cot \tfrac{1}{2}A = \sqrt{\frac{S(S-a)}{(S-b)(S-c)}},$$

and so cot $\frac{1}{2}$B and cot $\frac{1}{2}$C. We may also, if necessary, get sec $\frac{1}{2}$A, cosec $\frac{1}{2}$A, etc.

Since $\sin 2A = 2 \sin A \cos A$;

$$\therefore \sin A = 2 \sin \tfrac{1}{2}A \cdot \cos \tfrac{1}{2}A;$$

$$\therefore \sin A = 2 \cdot \sqrt{\frac{(S-b)(S-c)}{bc}} \times \sqrt{\frac{S(S-a)}{bc}};$$

$$\therefore \sin A = 2\frac{\sqrt{S(S-a)(S-b)(S-c)}}{bc}$$

$$= \frac{2}{bc}\sqrt{S(S-a)(S-b)(S-c)}.$$

Of course sin B and sin C have corresponding values, the only difference being in the quantity outside the root sign; thus sin C will have $\frac{2}{ab}$ outside, and sin B will have $\frac{2}{ac}$ outside.

Many other useful formulæ may be deduced from the above.

It must be remembered that S in the above formulæ $=\frac{a+b+c}{2}$;

$$\therefore S-a = \frac{b+c-a}{2};$$

$$\therefore S-b = \frac{a+c-b}{2};$$

$$\therefore S-c = \frac{a+b-c}{2}.$$

It is often required to adopt a formula to logarithmic computation—that is, to arrange it so that logarithms can be used in the application of the formula. Thus the formula

$$2 \sin^2 \tfrac{1}{2}A = -\frac{b^2+c^2-a^2}{2bc}$$

is not adapted to logarithmic calculation; but when the right-hand side is simplified, we get

$$2 \sin^2 \tfrac{1}{2}A = \frac{(a+c-b)\,(a+b-c)}{2bc},$$

which is adopted, for all the quantities in the formula are arranged as products or quotients, and it may be thus expressed:

$$\text{Log } 2 + 2 \log \sin \tfrac{1}{2}A = \log (a+c-b) + \log (a+b-c) - \log 2 - \log b - \log c,$$

from which $\log \sin \tfrac{1}{2}A = \tfrac{1}{2}\{\log (a+c-b) + \log (a+b-c) - 2 \log 2 - \log b - \log c\}$.

SOLUTION OF TRIANGLES.

The equality of triangles is treated upon by Euclid in the following propositions:—I. 4, 8, 26. Prop. I. 22 enables us to construct a triangle whose sides shall be equal to three given lines.

Props. II. 12 and 13 are the foundation of some of the most important formulæ used in trigonometry; and Prop. 7 of Book VI. points out to us what is generally known in trigonometry as the 'ambiguous case.'

The greater number of formulæ in trigonometry are derived from the peculiarities of similar figures, which are treated in the sixth book of Euclid; and it is very important indeed that this book should be carefully studied, in order that all the properties of similar figures, upon which so much in trigonometry depends, may be familiar.

The formulæ expressing the relation between the sides and the sines of the angles of a triangle will enable us to solve all the cases of right-angled triangles except one, viz. that case in which two sides are given, and the functions of the two oblique angles are required. In this case we shall employ Euclid I. 47 to find the third side. Thus, if a, b, c are the sides respectively opposite the angles A, B, C, and C is the right angle, we shall have

$$c = \sqrt{a^2+b^2}.$$
$$b = \sqrt{c^2-a^2}.$$
$$a = \sqrt{c^2-b^2}.$$

We may also notice that if C is the right angle, $A+B=90$, by Euclid I. 32;

$$\therefore A=90-B;$$
$$\therefore \sin A=\sin(90-B)=\cos B,$$
$$\text{and } \cos A=\sin B.$$

Similarly, $\tan A=\cot B$, etc.

Again, since in any triangle $\frac{\sin A}{\sin B}=\frac{a}{b}$, and in a right-angled triangle $\sin B=\cos A$,

$$\therefore \frac{\sin A}{\cos A}=\frac{a}{b};$$

$$\therefore \tan A=\frac{a}{b} \text{ in a right-angled triangle.}$$

Similarly, $\tan B=\frac{b}{a}$.

It must be noted that, since C is the right angle, $\sin C=1$, $\cos C=0$;

$$\therefore \tan C=\frac{1}{0}=\text{infinity},$$
$$\cot C=\frac{0}{1}=0.$$

We now take the several cases of right-angled triangles.

I. a and b given.

$$\text{Tan } A=\frac{a}{b}, \text{ or } \log\tan A=\log a-\log b+10,$$

which gives A.

$$c=\sqrt{a^2+b^2}, \text{ which gives } c.$$
$$B=90-A, \text{ which gives B.}$$

II. a and B given.

$$A=90-B, \text{ which gives A.}$$

$$\text{Tan } B=\frac{b}{a}, \text{ or } \log b=\log\tan B-10+\log a,$$

which give b.

$$c=\sqrt{a^2+b^2}, \text{ which gives } c.$$

III. B and c given.

$$A=90-B,$$
$$b=c\sin B,$$
$$a=c\sin A,$$

which give the remaining parts.

IV. a and c given.

$$b=\sqrt{c^2-a^2}=\sqrt{(c+a)(c-a)};$$

$$\therefore \log b=\tfrac{1}{2}\{\log(c+a)+\log(c-a)\}, \text{ which gives } b.$$

$$\text{Tan B}=\frac{b}{a}\text{ gives B.}$$

$$\text{Tan A}=\frac{a}{b},\text{ or A}=90-\text{B, give A.}$$

v. A and a given.

$$B=90-A\text{ gives B.}$$

$$b=a\frac{\sin B}{\sin A},\text{ or }\log b=\log a+\log\sin B-\log\sin A.$$

$$c=\sqrt{a^2+b^2}.$$

When B and b are given, the solution is similar to v.

" A and c	"	"	III.
" A and b	"	"	II.
" b and c	"	"	IV.

Proofs of certain identities which are only true in cases of right-angled triangles are sometimes given. The following will illustrate this class of examples.

Ex. 1. If ABC be a triangle right-angled at C, show that $\cos(2A-B)=3\left(\frac{a}{c}\right)-4\left(\frac{a}{c}\right)^3$.

$$\text{Cos}(2A-B)=\cos 2A\cos B+\sin 2A\sin B.$$

$$\text{Cos A}=\sin B,\text{ and }\sin A=\cos B;$$

$$\therefore\cos(2A-B)=\cos 2A\sin A+\sin 2A\cos A.$$

$$\text{But }\sin A=\frac{a}{c};\ \therefore\cos A=\sqrt{1-\frac{a^2}{c^2}};\ \therefore\cos^2 A=\frac{c^2-a^2}{c^2}.$$

$$\text{Cos }2a=2\cos^2 A-1=2\,.\frac{c^2-a^2}{c^2}-1=\frac{c^2-2a^2}{c^2}.$$

$$\text{Sin }2A=2\sin A\cos A=2\,.\frac{a}{c}\sqrt{\frac{c^2-a^2}{c^2}}$$

$$\therefore\cos(2A-B)=\frac{c^2-2a^2}{c^2}\cdot\frac{a}{c}+2\,.\frac{a}{c}\sqrt{\frac{c^2-a^2}{c^2}}\sqrt{\frac{c^2-a^2}{c^2}}$$

$$=\frac{a}{c^3}\{c^2-2a^2+2(c^2-a^2)\}$$

$$=\frac{a}{c^3}(3c^2-4a^2)$$

$$=3\left(\frac{a}{c}\right)-4\left(\frac{a}{c}\right)^3.$$

Ex. 2. Show that the area of a right-angled triangle $=S(S-c)$.

The area of a right-angled triangle $=\frac{ab}{2}$. We have therefore to show that $S(S-c)=\frac{ab}{2}$.

$$S=\frac{a+b+c}{2},$$

$$S-c=\frac{a+b-c}{2};$$

$$\therefore S(S-c)=\frac{(a+b+c)(a+b-c)}{4}$$

$$=\frac{a^2+b^2+2ab-c^2}{4}.$$

But $c^2=a^2+b^2$;

$$\therefore S(S-c)=\frac{c^2+2ab-c^2}{4}$$

$$=\frac{2ab}{4}=\frac{ab}{2};$$

$\therefore S(S-c)=$ area of a right-angled triangle.

Ex. 3. Given q the perimeter of a right-angled triangle, and p the perpendicular from C on the hypotenuse, to find a, b, c.

Let CD be the perpendicular p.

Then $a+b+c=q$,

and $c^2=a^2+b^2$.

Also $c : b :: a : p$ by similar triangles;

$$\therefore p=\frac{ab}{c};$$

$$\therefore p^2=\frac{a^2b^2}{c^2}=\frac{a^2b^2}{a^2+b^2},$$

and $q=a+b+\sqrt{a^2+b^2}$,

from which the values of a, b, and then c, may be found.

Ex. 4. If a triangle is right-angled at C, show that

$$\cot\frac{A}{2}=\frac{c+b}{a}.$$

$$a^2=c^2-b^2=(c+b)(c-b);$$

$$\therefore c+b=\frac{a^2}{c-b}, \text{ and } \frac{1}{c-b}=\frac{c+b}{a^2};$$

$$\therefore \frac{c+b}{c-b}=\left(\frac{c+b}{a}\right)^2;$$

$$\therefore \frac{1+\frac{b}{c}}{1-\frac{b}{c}}=\left(\frac{c+b}{a}\right)^2$$

$$\text{But } \frac{b}{c}=\cos A;$$

$$\therefore \frac{1+\cos A}{1-\cos A}=\left(\frac{c+b}{a}\right)^2.$$

$$\text{But } \cot^2 \tfrac{1}{2}A=\frac{1+\cos A}{1-\cos A};$$

$$\therefore \cot \tfrac{1}{2}A=\frac{c+b}{a}.$$

Ex. 5. Given in a right-angled triangle, C being the right angle—

Angle $A=53^\circ\ 8'$, of which log tan $=10\cdot1249818$; AC $=288$ feet, of which the log is $2\cdot4593925$; and that log $382\cdot457=2\cdot5842743$. Find BC.

$$\text{Cos } A=\frac{AC}{AB};\ \therefore AB=\frac{288}{\cos A}.$$

$$\text{Sin } A=\frac{BC}{AB};\ \therefore BC=AB \sin A$$

$$=288\ \frac{\sin A}{\cos A}=288 \tan A;$$

$$\begin{aligned}\therefore \log BC&=\log 288+\log \tan A-10\\&=\log 288+\log \tan 53^\circ\ 8'-10\\&=2\cdot4593925+\cdot1249818\\&=2\cdot5842743\\&=\log 382\cdot457;\end{aligned}$$

$$\therefore BC=382\cdot457 \text{ feet.}$$

Ex. 6. If ABC be a triangle right-angled at C, show that

$$\sin 2A+\cos 2A+\tan 2A=\frac{(a+b)^2-2a^2}{a^2+b^2}-\frac{2ab}{a^2-b^2},$$

$$\sin A=\frac{a}{c},$$

$$\sin B=\cos A=\frac{b}{c};$$

$$\therefore 2 \sin A \cos A=\frac{2ab}{c^2};\ \therefore \sin 2A=\frac{2ab}{c^2},$$

$$\cos 2A = 2\cos^2 A - 1 = 2\cdot\frac{b^2}{c^2} - 1 = \frac{2b^2 - c^2}{c^2},$$

$$\tan 2A = \frac{\sin 2A}{\cos 2A} = \frac{2ab}{2b^2 - c^2};$$

$$\therefore \sin 2A + \cos 2A + \tan 2A = \frac{2ab}{c^2} + \frac{2b^2 - c^2}{c^2} + \frac{2ab}{2b^2 - c^2}$$

$$= \frac{2ab}{a^2+b^2} + \frac{2b^2 - (a^2+b^2)}{a^2+b^2} + \frac{2ab}{2b^2 - (a^2+b^2)}$$

$$= \frac{2ab + b^2 - a^2}{a^2+b^2} + \frac{2ab}{b^2 - a^2}$$

$$= \frac{(a+b)^2 - 2a^2}{a^2+b^2} - \frac{2ab}{a^2 - b^2}.$$

Ex. 7. ABC is half an equilateral triangle, C being the right angle and A an angle of 60°. A straight line is drawn from the angle B to the middle of the opposite side. Find the ratio of the sines of the angles into which the angle B is divided.

Let D be the middle point of AC, and a the hypotenuse.

$$\text{Then } AC = \frac{a}{2},\ AD = DC = \frac{a}{4},\ BC = \frac{a\sqrt{3}}{2},$$

$$BD = \sqrt{\frac{a^2}{16} + \frac{3a^2}{4}} = \frac{a}{4}\sqrt{13}.$$

$$\text{Then } \sin A : \sin ABD :: \frac{a}{4}\sqrt{13} : \frac{a}{4},$$

$$\text{or } \frac{\sqrt{3}}{2} : \sin ABD :: \sqrt{13} : 1;$$

$$\therefore \sin ABD = \frac{\sqrt{3}}{2\sqrt{13}}.$$

$$\text{Again, } \sin C : \sin DBC :: \frac{a}{4}\sqrt{13} : \frac{a}{4};$$

$$\therefore 1 : \sin DBC :: \sqrt{13} : 1;$$

$$\therefore \sin DBC = \frac{1}{\sqrt{13}};$$

$$\therefore \sin ABD : \sin DBC :: \frac{\sqrt{3}}{2\sqrt{13}} : \frac{\sqrt{1}}{\sqrt{13}} :: \frac{\sqrt{3}}{2} : 1$$

$$:: \sqrt{3} : 2.$$

OBLIQUE-ANGLED TRIANGLES.

In the solution of these triangles five different sets of data may be given. Every set must have three parts of the triangle included in it, and any three parts will be sufficient except three angles; if three angles only are given, the sides cannot be calculated, for the magnitude of the angles has no connection with the magnitude of the triangle as a whole.

Let ABC be a triangle, of which A, B, and C represent the angles, and a, b, c the sides respectively opposite to A, B, and C. We may then have the following varieties in the data given:—

I. Two angles and a side between them—that is, A, C, b; A, B, c; and B, C, a.

II. Two angles and a side opposite one of them, viz. A, C, a; A, C, c; B, C, b; B, C, c; A, B, a; A, B, b.

III. Two sides and the angle between them—c, A, b; a, C, b; c, B, a.

IV. Two sides and an angle opposite one of them—a, b, A; a, b, B; b, c, B; b, c, C; c, a, C; c, a, A.

V. All three sides, a, b, c, given.

It will only be necessary to show how the other parts are obtained in one set from each case.

I. A, C, b given, to find a, c, B.

$$B = 180 - (A + C) \text{ gives B},$$

$$a = b\,\frac{\sin A}{\sin B} \text{ gives } a,$$

$$c = b\,\frac{\sin C}{\sin B} \text{ gives } c.$$

Or by logs,

$$\log a = \log b + \log \sin A - \log \sin B,$$
$$\log c = \log b + \log \sin C - \log \sin B$$

II. A, C, a given, to find b, c, B.

The process is similar to the last.

III. c, A, b given, to find C, a, B.

Several methods are given for solving this case, but the following is the easiest to remember:—

$$\text{Cos A}=\frac{b^2+c^2-a^2}{bc}.\quad \text{From this we get } a.$$

$$\text{Sin B}=\frac{b.\sin A}{a},\ \text{which gives B.}$$

$$C=180-(A+B),\ \text{which gives C.}$$

We advise the student to make a special study of this case, and to master the method of solution by employing a subsidiary angle.

IV. a, b, A given, to find C, c, B.

$$\text{Sin B}=\frac{b\sin A}{a},\ \text{which gives B.}$$

$$C=180-(A+B),\ \text{which gives C.}$$

$$c=a\,\frac{\sin C}{\sin A},\ \text{which gives } c.$$

V. a, b, c given, to find A, B, C.

There are many formulæ suitable for finding the functions of the angles when the sides are given; for example,

$$\sin A=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)},\ \text{which gives A.}$$

B and C may be found by the same formula, or by the method of Case III.

When all the sides are given, we may also find the angles by dividing the given triangle into two by a perpendicular from the vertex on the base, thus:

Let ABC be the triangle, BD a perpendicular from B on AC.

$$\text{Then } BD^2=AB^2-AD^2=BC^2-CD^2.$$

$$\text{Let } x=AD.$$

$$\begin{aligned}\text{Then } c^2-x^2&=a^2-(c-x)^2\\&=a^2-c^2+2cx-x^2;\end{aligned}$$

$$\therefore\ 2cx=2c^2-a^2;$$

$$\therefore\ x=\frac{2c^2-a^2}{2c};$$

$$\therefore\ c-x \text{ or } CD=c-\frac{2c^2-a^2}{2c}=\frac{a^2}{2c}.$$

We have now all the necessary data for solving the two right-angled triangles ABD and BDC, for two sides are given, thus:

$$AB = C,\ AD = \frac{2c^2 - a^2}{2c},\ \text{and } CD = \frac{a^2}{2c},\ \text{and } BC = a;$$

$$\therefore \cos A = \frac{AD}{AB} = \frac{2c^2 - a^2}{2c^2},$$

$$\cos C = \frac{DC}{BC} = \frac{a^2}{2ac} = \frac{a}{2c},$$

$$\text{and } B = 180 - (A + C).$$

When two sides and an included angle are given, it is best to work out the other angles by the formulæ—

$$\tan \tfrac{1}{2}(B + C) = \tan \tfrac{1}{2}(180 - A) = \cot \frac{A}{2},$$

$$\tan \tfrac{1}{2}(B - C) = \frac{b - c}{b + c} \cot \frac{A}{2}.$$

In this way we get $B + C$ and $B - C$, from which B and C may be found.

Certain interesting identities may be proved from the formulæ for oblique triangles, and whenever an identity is required to be proved, it must be understood that the formulæ for oblique triangles only is to be used, unless it is stated that the triangle in question is to be taken as right-angled. In oblique-angled triangles, sin C, instead of being unity,

$$= \frac{C \sin A}{a} = \frac{c \sin B}{b}.$$

We conclude this part with an example which has been taken from examination papers on oblique and other triangles.

Ex. 8. In any triangle, $\sin A - \sin B$ is less than $\sin C$.

As any two sides of a triangle are greater than the third side,

$$\therefore b + c > a;$$

$$\therefore c > a - b;$$

$$\therefore 1 > \frac{a - b}{c} > \frac{a}{c} - \frac{b}{c}.$$

$$\text{But } \frac{a}{c} = \frac{\sin A}{\sin C},\ \text{and } \frac{b}{c} = \frac{\sin B}{\sin C};$$

$$\therefore \frac{a}{c} - \frac{b}{c} = \frac{\sin A}{\sin C} - \frac{\sin B}{\sin C}$$

$$= \frac{\sin A - \sin B}{\sin C}.$$

But it has been shown that 1 is $> \frac{a}{c} - \frac{b}{c}$;

$$\therefore 1 > \frac{\sin A - \sin B}{\sin C};$$

$$\therefore \sin C > \sin A - \sin B,$$
$$\text{or } \sin A - \sin B < \sin C.$$

HEIGHTS AND DISTANCES OF OBJECTS.

Many simple methods have been devised by which the heights and distances of inaccessible objects may be ascertained approximately without the use of expensive instruments. We shall first give a few examples of calculations of this kind, and then show the methods of calculation when delicate apparatus is applied.

Ex. 1. A man has a pole 12 feet high, and a plumb line. He wishes to ascertain the height of a tower whose shadow on a horizontal plane is 132 feet long from the centre of the base.

The man fixes the pole in a vertical position, and finds its shadow is $9\frac{1}{2}$ feet long. What is the height of the tower?

This example is simple enough. The triangles formed by the pole, the shadow, and the distance from the top of the pole to the point of the shadow will be similar to the triangle formed by the tower, its shadow, and the distance from the top of the tower to the limit of its shadow. They will be both right-angled triangles; and length of pole's shadow : length of tower's shadow :: length of pole : height of tower;

$$\therefore 9\tfrac{1}{2} \text{ ft.} : 132 \text{ ft.} :: 12 : x; \qquad \text{(Euc. VI. 4.)}$$

$$\therefore \frac{19x}{2} = 12 \times 132,$$

$$19x = 24 \times 132,$$

$$x = \frac{24 \times 132}{19} = 166\tfrac{3}{4} \text{ feet nearly.}$$

This method can be employed when the shadows do not fall on a plane which is perfectly horizontal, provided that the slope from the horizontal is continuous.

The height of an object which stands on an irregular surface may be found by means of a spirit level and three poles, as follows:—

Let AB be the tower, FDB the irregular surface upon which it stands. Place one pole EF at F, and the second pole CD at D. Let the top of the pole EF, and the point C in the pole CD, be in a straight line with A, the top of the tower.

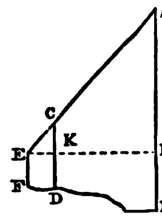

Fig. 18.

Place the third pole on the top of EF, and by means of the spirit level adjust it to a horizontal position.

Mark the point on CD where this pole crosses it when horizontal, and measure the distance from that point to C and E respectively. The point with EC will make a triangle similar to EAH; and if K be the point on CD, and H the point in the same horizontal line on the tower,

$$EK : EH :: KC : AH;$$

$$\therefore AH = \frac{EH.KC}{EK},$$

and $\frac{EH.KC}{EK} + HB$ will be the height of the tower.

Ex. 2. A man standing on the side of a river observes that the reflection of the top of a tower on the other side is seen by himself at a point 29 yards from the bank on which he stands. The river is known to be 1100 feet wide, and his eye is 4 feet higher than the water. What is the height of the tower?

Let BE be the tower, DE the width of the river, AD the height of the eye, C the point at which the top of the tower is reflected.

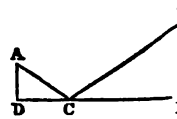

Fig. 19.

Then AD = 4 feet, DC = 87 feet, DE = 1100 feet.

The triangles ADC, BCE are similar, for the angle BCE = angle ACD, and ADC, BEC are right angles;

$$\therefore DC : AD :: CE : BE;$$

$$\therefore 87 : 4 :: (1100 - 87) : BE;$$

$$87 : 4 :: 1013 : \text{BE};$$

$$\therefore \text{BE} = \frac{4 \times 1013}{87} = 46{\cdot}57 \text{ feet.}$$

By means of a chain and pegs, the distance of an inaccessible object may be ascertained.

Let C be the inaccessible object.

Measure a base line AB, say a chain long. Put a peg at each end, A and B.

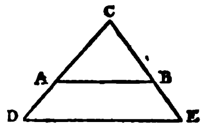

Fig. 20.

Take a point D in the same line as A and C. Measure AD. Walk in a line DE parallel to AB. Let E be a point in line with BC. Measure DE and EB.

Let x be the distance AC, y the distance CB.

Let AD $= a$, DE $= b$, AB $= c$, BE $= d$.

$$\text{Then DE} : \text{AB} :: \text{DC} : \text{AC},$$
$$\text{or} :: \text{CE} : \text{BC};$$
$$\therefore b : c :: x + a : x,$$
$$:: y + d : y.$$

From the first, $c(x + a) = bx$;

$$\therefore cx + ac = bx;$$
$$\therefore x(b - c) = ac;$$
$$\therefore x = \frac{ac}{b - c}.$$

From the second, $by = c(y + d)$;

$$\therefore y = \frac{cd}{b - c};$$

$$\therefore \text{the distance of C from A is } \frac{\text{AD} . \text{AB}}{\text{DE} - \text{AB}},$$

$$\text{and the distance of C from B is } \frac{\text{AB} . \text{BE}}{\text{DE} - \text{AB}}.$$

A better method with a chain and pegs is to take any line from D, and instead of making DE parallel to AB, measure AE and BD.

The following example, taken from the B.A. examination papers, illustrates the method, which involves a knowledge of trigonometry.

Ex. 3. A traveller, unprovided with any instrument for measuring angles, arrives at a river and wishes to know his distance from an object P on the opposite bank. Accordingly he measures a base line AB, a feet in length, and from A and B to distances AC, BD directly in a

line with and away from P, c and e feet in length respectively; finally, he finds by measurement the distances of C from B, and D from A, to be h and k feet respectively. Find by trigonometry PA, PB, and the perpendicular distance of P from AB.

Let P be the object, and A, B, C, D, the points as given in the example.

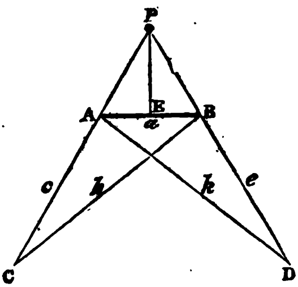

Fig. 21.

Then

$AC=c$, $BC=h$, $AD=k$.

$BD=e$, $AB=a$.

And PAC, PBD are straight lines.

In the triangle ABC, all the sides are given;

$$\therefore \cos ABC=\frac{a^2+c^2-h^2}{2ac}.$$

This gives the angle BAC;

$\therefore$ $180-BAC=PAB$.

Similarly,

$$\cos ABD=\frac{a^2+e^2-k^2}{2ae}.$$

This gives angle ABD, and $180-ABD=ABP$.

Now, in the triangle PAB, we have two angles, PAB, PBA, and a, a side between them. Let $PAB=\theta$, $PBA=\phi$.

Then $\angle$ APB $=180-(\theta+\phi)$, which gives $\angle$ APB.

Let $\angle$ APB $=\beta$.

Then $\frac{PA}{a}=\frac{\sin\phi}{\sin\beta}$; $\therefore$ $PA=a.\frac{\sin\phi}{\sin\beta}$, which gives PA.

Similarly, $\frac{PB}{a}=\frac{\sin\theta}{\sin\beta}$; $\therefore$ $PB=a.\frac{\sin\theta}{\sin\beta}$, which gives PB.

Now, to find PE, the perpendicular on AB from P.

$$\frac{PE}{PA}=\sin\theta;\ \therefore\ PE=PA.\sin\theta.$$

In many cases the horizontal distance of an object cannot be measured, so that both the summit and foot are inaccessible. The following example illustrates the method of calculating the height in such a case with a chain and poles.

AB is a tower whose foot is inaccessible. A point C is taken, and a pole 5 feet long is erected. Another pole 12 feet long is taken in a line with the tower, and

placed so that its top is in a line with the top of the first pole and the top of the tower. Let D be the point at which it is placed.

CD is measured, and is found to be 13 feet. Another point E is selected in the line BC, and the poles are adjusted as before. It is found that CE = 150 feet, and the poles are $4\frac{1}{2}$ feet apart. It is required to find the height of the tower from C, and its distance from these data.

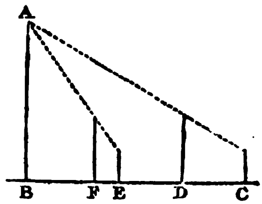

Fig. 22.

Now, suppose that a line were drawn through the tops of the two shorter posts to the tower. This line would be 5 feet above the ground, and would form a triangle with each pair of the poles, whose dimensions would be as follows:—1st, base = DE = 13 feet; altitude = 12 − 5, or 7 feet.

This triangle would be similar to the whole triangle formed by the line, and the height of the tower above 5 feet from the ground. Let h = height of tower above this point, b the base from the top of the pole to the tower.

$$\text{Then } h : b :: 7 \text{ feet} : 13 \text{ feet.}$$

Similarly, from the other triangle,

$$h : (b-150) :: 7 \text{ feet} : 4\tfrac{1}{2} \text{ feet.}$$

$$\text{From the first, } h = \frac{7}{13}b.$$

$$\text{From the second, } h = \frac{7(b-150)}{4\frac{1}{2}} = \frac{14(b-150)}{9};$$

$$\therefore \frac{7}{13}b = \frac{14(b-150)}{9};$$

$$\therefore 63b = 182b - 27300;$$

$$\therefore b = \frac{27300}{119} \text{ feet} = 221\frac{1}{119} \text{ feet};$$

$$\therefore h = \frac{7}{13} \times \frac{27300}{119} = 123\frac{9}{17}.$$

Adding the length of short pole,

$$h = 128\frac{9}{17} \text{ feet.}$$

The above methods do not involve angular observations. To measure angles with great accuracy, it is

necessary to use a theodolite with Vernier scale. The construction of these apparatus, together with Hadley's sextant, should be carefully studied, and an account of their construction and use will be found in almost any work on trigonometry.

A useful instrument for measuring angles of elevation or depression when great accuracy is not required, may be constructed by the student himself. It is called a quadrant. It consists of a piece of thin wood in the form of a quadrant of a circle of about 10 inches radius. Great care must be taken that the edges which meet at the centre of the circle are exactly at right angles. A small pivot is placed at the point which forms the centre of the circle, and the pivot is fixed at right angles to the plane of the quadrant. On the other side of the quadrant at the same point a pin is fixed, or a hole made through it so that a plumb line can be attached. The quadrant is then divided near its curved edge into 90 divisions for degrees, and, if possible, each one subdivided to show quarters or tenths of a degree.

On one edge two small bits of brass, each having a small hole at equal distances from the edge, are fastened in. The quadrant is attached by the pivot to an upright pole of a convenient height for the eye to be on a level with the edge.

E, E′ shows the position of these brass attachments. When this apparatus is used, say, to find the elevation of a star, the pole is fixed firmly in the ground. The side E, E′ is turned towards the object, and the quadrant put in a vertical plane. The quadrant is then turned on its pivot until the object is seen through both the holes in E, E′; the division under the plumb line is then read off, and the number of degrees between the plumb line and A is the required elevation.

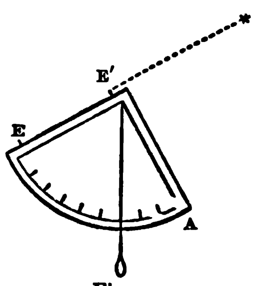

Fig. 23.

With this instrument and a chain, many very interesting problems may be worked out.

Ex. 4. A person observes the elevation of a tower to

be 60°, and on receding from it 100 yards he finds its elevation to be 30°. Find the height of the tower.

Let PR be the height of the tower, P being its summit. Let F be the point at which the first observation was made, E the point at which the second observation was made.

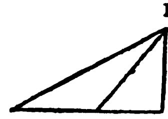

Fig. 24.

Join EP, FP.

Then angle PFR = 60°, angle PER = 30°, and EF = 100 yards.

Since the angle PFR = 60°, the exterior angle PFE = 120°;

$$\therefore \text{ angle } EPF = 30^\circ;$$

$$\therefore EF = FP = 100.$$

$$\text{But } \frac{PR}{PF} = \sin 60;$$

$$\therefore PR = PF \sin 60$$

$$= 100\frac{\sqrt{3}}{2} = 50\sqrt{3} \text{ yards};$$

$\therefore$ the height of the tower is $3 \times 50\sqrt{3}$ feet $= 150\sqrt{3}$ feet. This result, of course, does not include the height of the eye above the ground at the time of the observations. If this is 5 feet, the tower will be higher by 5 feet.

Ex. 5. From the top of a rock 50 feet above the water level of a river, I observe a castle on the cliff on the opposite side. On the top of this rock, 5 feet above the ground, I find the angle of elevation of the top of the castle to be 10°, and the angle of depression of the surface of the water to be 14°. Given tan 10° = ·1763, tan 14° = ·2492, find the height of the castle above the water level.

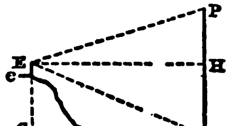

Fig. 25.

PEH = angle of elevation = 10°,
HER = angle of depression = 14°,
EG = 55 feet;
$\therefore$ HR = 55 feet.

$$\frac{RH}{EH} = \tan HER;$$

$$\therefore \frac{55}{EH} = \tan 14;$$

$$\therefore EH = \frac{55}{\tan 14},$$

$$\frac{PH}{EH} = \tan 10;$$

$$\therefore \text{PH} = \text{EH} \tan 10$$
$$= \frac{55 \cdot \tan 10}{\tan 14} = \frac{55 \times \cdot 1763}{\cdot 2492} = 45{\cdot}8 \text{ feet};$$

$\therefore$ the castle is $55 + 45{\cdot}8$, or $100{\cdot}8$ feet above the water.

In taking an angle of depression the quadrant is used, but the sight is taken downwards along E′, E.

The angles taken by the quadrant are angles subtended by objects that are vertically upwards; but by using the other instruments above referred to, angles may be taken which are subtended between points which are either horizontal or oblique.

The following examples show how oblique or horizontal angles are used:—

Let PR be a tower, and E a point at which the observer stands. Let a mark be placed at E, and let the observer move away to F, and measure the angles PFR, PFE, and the distance EF.

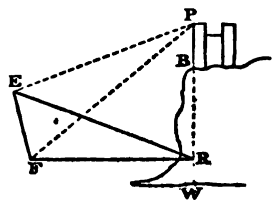

Fig. 26.

At E let him measure the angle PEF. These data will be sufficient to find PR.

Let angle PFR $= \theta$,
angle PFE $= \phi$,
angle PEF $= \beta$,
and the distance EF $= a$.

PF is first determined, thus:

$$\frac{\text{PF}}{\text{EF}} = \frac{\text{PF}}{a} = \frac{\sin \beta}{\sin \text{EPF}} = \frac{\sin \beta}{\sin (180 - \theta - \phi)} = \frac{\sin \beta}{\sin (\theta + \phi)};$$

$$\therefore \text{PF} = a \cdot \frac{\sin \beta}{\sin (\theta + \phi)}, \text{ which gives PF.}$$

$$\text{And PR} = \text{PF} \cdot \sin \theta = a \cdot \frac{\sin \theta \cdot \sin \beta}{\sin (\theta + \phi)};$$

$\therefore \log \text{PR} = \log a + \log \sin \theta + \log \sin \beta - \log \sin (\theta + \phi) - 10.$

In this case it is not necessary that E should be in the same horizontal plane with F.

Ex. 6. The angular elevation of a tower at a place E due south of it is $30°$, and at a place due west of A, and at a distance of 200 feet from it, is $18°$. Find the height of the tower, having given $\sin 30° = \cdot 5$, $\sin 18° = \cdot 309017$, $\sin 72° = \cdot 951057$.

Let F in the preceding figure be the position at which the last observation was taken, and PR the tower.

$$\begin{aligned} \text{Then angle } PFR &= 18^\circ, \\ \text{angle } FPR &= 72^\circ, \\ \text{angle } PER &= 30^\circ, \\ \text{angle } EPR &= 60^\circ, \\ \text{angle } EF &= 200 \text{ feet.} \end{aligned}$$

Let x = height of tower in feet;

$$\therefore x : ER :: \sin 30^\circ : \sin 60^\circ;$$

$$\therefore ER = x.\sqrt{3}.$$

Again, $x : FR :: \sin 18^\circ : \sin 72^\circ$;

$$\therefore FR = x\frac{\sin 72^\circ}{\sin 18^\circ}.$$

But F is due west from E; $\therefore$ FER is a right angle;

$$\therefore FR^2 = EF^2 + ER^2;$$

$$\therefore x^2\left(\frac{\sin 72^\circ}{\sin 18^\circ}\right)^2 = 200^2 + 3x^2;$$

$$\therefore x^2\left(\frac{\cdot 951057}{\cdot 309017}\right)^2 - 3x^2 = 200^2;$$

$$\therefore x^2(9\cdot 467929 - 3) = 200^2,$$

$$x^2 = \frac{200^2}{6\cdot 467929},$$

$$x = 78\cdot 6 \text{ feet nearly.}$$

Ex. 7. A tower of cylindrical form has a flagstaff on the top, at the end of its axis. The thickness of the tower is m feet, and at a distance of n feet from the foot of the tower, the angles of elevation of the top of the flagstaff and tower are ϕ and θ respectively. Find the height of the tower and the flagstaff.

Let C be the top of the flagstaff, E the bottom of the axis of the tower. Let D be the edge of the top of the tower, and B the bottom of the tower and the outside of it.

Let A be the point at which the observations are taken.

$$\text{Then } \frac{BD}{AB} = \tan \theta;$$

$$\therefore BD = AB \tan \theta = n \tan \theta,$$

which is the height of the tower.

$$AE = AB + BE = n + \frac{m}{2} = \frac{2n+m}{2},$$

$$\frac{EC}{AE} = \tan\phi \,;\ \therefore EC = AE \tan\phi\,;$$

$$\therefore EC = \frac{2n+m}{2} \tan\phi,$$

which is the height of the top of the flagstaff from the ground.

The following are the methods of calculating the distance between inaccessible objects which are sometimes used.

Let E, P be two inaccessible points (see fig. on page 154). Take two points F and R, each visible from the other.

Measure FR. Let it $= a$ feet. At F measure angles EFR, PFR, and at R the angles PRF, ERF.

Let $EFR = \alpha$, $ERF = \beta$, $PFR = \theta$, $PRF = \phi$.

$$\text{Then } FER = 180 - EFR - ERF,$$
$$FPR = 180 - PFR - PRF.$$

$$\text{Then } EP^2 = a^2 \left\{ \frac{\sin^2\beta}{\sin^2(\alpha+\beta)} + \frac{\sin^2\phi}{\sin^2(\theta+\phi)} - \frac{2\sin\phi\sin\beta\cos(\alpha-\theta)}{\sin(\alpha+\beta)\sin(\theta+\phi)} \right\}.$$

A proof of this formula may be found in Garnet's *Trigonometry*, published by Stewart, p. 120.

If the distances between three points are known, their distance from a point P may be determined (see Todhunter's *Trigonometry*, Art. 242).

There are many different ways of finding the distance of inaccessible objects besides those given. Practice alone can enable the student in all cases to select the best method.

AREAS OF TRIANGLES, ETC.

The following are the most important formulæ relating to the areas of triangles, etc., and as their proofs are easily obtained, we leave them for the student to get up from his text book.

1. All the sides given. $\text{Area} = \sqrt{S(S-a)(S-b)(S-c)}$.
2. One side and the two adjacent angles.

$$\text{Area} = \frac{a^2}{2} \cdot \frac{\sin B \cdot \sin C}{\sin(B+C)}.$$

3. Two sides and the included angle.

$$\text{Area} = \tfrac{1}{2}\, ac \sin B = \tfrac{1}{2}\, bc \sin A = \tfrac{1}{2}\, ab \sin c.$$

4. One side and three angles.

$$\text{Area} = \frac{b^2 \sin A \cdot \sin C}{2 \sin B}.$$

If two triangles have their bases put together so as to form a quadrilateral figure, and the opposite angles of this quadrilateral figure are supplements to each other, then area of quadrilateral figure $= \sqrt{(S-a)(S-b)(S-c)(S-d)}$ where $S = \frac{1}{2}(a+b+c+d)$.

If ABC is a right-angled triangle, and C the right angle, also a, b, c the sides respectively opposite A, B, C, then area $= S(S-c)$.

There are several other forms in which the area of a triangle can be expressed. The following examples are selected from the examination papers:—

Ex. 8. The sides of a triangle are 13, 14, and 15 feet; find its area.

$$\text{Area} = \sqrt{S(S-a)(S-b)(S-c)}.$$

$$S = 21;$$

$$\therefore \text{area} = \sqrt{21 \times 8 \times 7 \times 6}$$

$$= \sqrt{21 \times 8 \times 42} = \sqrt{42 \times 4 \times 42}$$

$$= 42 \times 2 = 84 \text{ feet.}$$

Ex. 9. A tower 100 feet high is observed at a station F, which is on a level with the base of the tower, and the angle of elevation of the top of the tower is found to be 45°. The observer then proceeds from F to E in a direction at right angles to the line joining F to the base of the tower; and he finds that at the station E (which is on the same level with F) the angle of elevation of the top of the tower is 30°. What is the distance between the stations E and F.

Let $x =$ distance EF (see last figure).

$$\frac{PR}{ER} = \tan PER = \tan 45^\circ;$$

$$\therefore PR = FR = \frac{PR}{\tan 45^\circ} = \frac{100}{1} = 100.$$

$$\frac{PR}{FR} = \tan PFR = \tan 30^\circ;$$

$$\therefore FR = \frac{PR}{\tan 30^\circ} = \frac{100}{\frac{1}{\sqrt{3}}} = 100\sqrt{3};$$

$\therefore$ since FER is a right-angled triangle, $FR^2 = EF^2 + ER^2$;

$$(100\sqrt{3})^2 = x^2 + 100^2;$$
$$\therefore x^2 = 100^2(3-1) = 100^2 \times 2;$$
$$\therefore x = 100\sqrt{2}.$$

Ex. 10. Two cliffs stand facing each other on opposite sides of a river. From the top of one of them, known to be 200 feet high, the angles of depression of the summit and foot of the other are observed to be 30° and 45° respectively. Find in feet and inches the height of the latter and the breadth of the river.

Let AB be the tower 200 feet high, CD the tower whose height is required, AE the horizontal line.

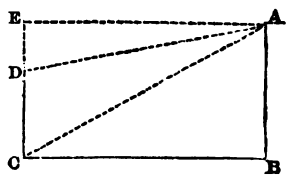

Fig. 27.

Then $\angle EAD = 30$, $EAC = 45$.

But if $EAC = 45$, $EA = AB = 200$; $\therefore BC = 200$.

Again, $\dfrac{DE}{AE} = \tan 30 = \dfrac{1}{\sqrt{3}}$;

$$\therefore DE = AE \tan 30 = AE \cdot \frac{1}{\sqrt{3}} = \frac{200}{\sqrt{3}}$$
$$= \frac{200}{3} \cdot \sqrt{3} = \frac{200 \times 1 \cdot 732}{3} = \frac{346 \cdot 400}{3} = 115 \cdot 466;$$
$$\therefore CD = 200 - 115 \cdot 466 \text{ feet} = 84 \cdot 534$$
$$= 84 \text{ feet } 6 \cdot 5 \text{ inches.}$$

Ex. 11. Given of a plane triangle, the base C, and the two base angles A, B. Find the two sides a and b, the altitude h, and the area Δ.

1. The sides a, b.

Find C from $C = 180 - (A + B)$.

$$\frac{\text{Sin } C}{c} = \frac{\sin A}{a}; \therefore a = c \cdot \frac{\sin A}{\sin C}.$$

Similarly, $b = \dfrac{c \cdot \sin b}{\sin C}$.

2. The altitude h.

$$\frac{CD}{BC} = \sin B;$$
$$\therefore CD = BC \sin B,$$
$$h = a \cdot \sin B.$$

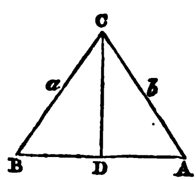

Fig. 28.

3. The area. This is given in a formula above.

$$\Delta = \frac{a^2}{2} \cdot \frac{\sin B . \sin C}{\sin (B+C)}.$$

Ex. 12. In any triangle ABC, the tangent of half the difference of the angles B and C is to the tangent of half their sum as the difference of the two sides AB and AC is to their sum, having given

$$\begin{cases} \log 2 = \cdot 3010300, \ \log \tan 35^\circ\ 49' = 9\cdot 8583357, \\ \log 3 = \cdot 4771213, \ \log \tan 35^\circ\ 49'\ 10'' = 9\cdot 8583800. \end{cases}$$

$$\frac{\text{Tan}\ \frac{1}{2}(B-C)}{\text{Tan}\ \frac{1}{2}(B+C)} = \frac{b-c}{b+c} = \frac{17-7}{17+7} = \frac{10}{24} = \frac{5}{12}.$$

But $\tan \frac{1}{2}(B+C) = \tan \frac{1}{2}(180 - A) = \cot \frac{1}{2}A$;

$$\therefore \tan \frac{1}{2}(B-C) = \frac{5}{12} . \cot \tfrac{1}{2}A$$

$$= \frac{5}{12} . \cot 30^\circ, \ \frac{5}{12} . \sqrt{3};$$

$$\therefore \log \tan \tfrac{1}{2}(B-C) = \log 5 + \tfrac{1}{2} \log 3 - \log 12 + 10$$
$$= 1 - \log 2 + \tfrac{1}{2} \log 3 - \log 3 - 2 \log 2 + 10$$
$$= 1 - \cdot 3010300 + \cdot 2385606 - \cdot 4771213 - 6020600 + 10$$

1·0000000	− ·3010300
·2385606	− ·4771213
10·0000000	− ·6020600
11·2385606	− 1·3802113
1·3802113	
9·8583493	

Subtracting the given log tangents, we find that for 10″ the difference is ·0000443.

Subtracting log tan $35^\circ\ 49'$ from our result above, we get ·0000036.

$\therefore$ The real value of tan of which the log is 9·8583493 is $35^\circ\ 49' + \frac{36}{443}$ of 10″.

$$\text{But } \frac{36}{443} \text{ of } 10'' = \frac{360}{443} = \cdot 85'';$$

$$\therefore \tfrac{1}{2}(B-C) = 35^\circ\ 49'\ 0''\cdot 85;$$

$$\therefore B - C = 71^\circ\ 39'\ 1''\cdot 7.$$

SUMMARY OF FORMULÆ,

WITH

HINTS ON THEIR PROOFS.

I. ARITHMETIC.

1. If a, b, c, d be the digits of a number, and r the radix, the number is ar^3+br^2+cr+d.

$$ar^3+br^2+cr+d=a(r^3-1)+b(r^2-1)+c(r-1)$$
$$+a+b+c+d,$$

which is divisible by $r-1$ if $a+b+c+d$ is divisible.

2. $$=a(r^3+1)+b(r^2-1)+c(r+1)+b+d-(a+c),$$

which is divisible by $r+1$ if $(b+d)-(a+c)$ is divisible.

3. Value of a pure circulating decimal $\cdot$MMM, M containing p figures recurring *ad infinitum.* S = value of equivalent vulgar fraction.

$S=\dfrac{M}{10^p-1}$. See p. 10.

4. Value of mixed circulating decimal $\cdot$PQQ, P containing m digits which do not recur, Q containing n digits which recur.

$S=\dfrac{PQ-P}{10^m(10^n-1)}$. See p. 11.

PQ in this case does not represent the product of the n digits and the m digits, but these digits placed after each other in the form of a number.

Rule of three. If $a : b :: c : x$,

5. $$x=\frac{bc}{a}.$$

6. $$\text{Log }(N+1)=\log N+2\mu\left\{\frac{1}{2N+1}+\frac{1}{3}\cdot\frac{1}{(2N+1)^3}+\frac{1}{5}\cdot\frac{1}{(2N+1)^5}+\text{etc.}\right\}.$$

$$\mu=\cdot 4342944\text{8}.$$

7. $\text{Log } mn = \log m + \log n.$

$$\text{Let } p = \log_a m,\ q = \log_a n;$$
$$\therefore\ m = a^p,\ n = a^q;$$
$$\therefore\ mn = a^{p+q};$$
$$\therefore\ p + q = \log_a (mn) = \log_a m + \log_a n.$$

8. $\text{Log } \frac{m}{n} = \log m - \log n.$ Proof similar to above.

9. $\text{Log}_a(m^r) = r \log_a m.$

10. $\text{Log}_{10}(\text{N} \times 10^n) = \log_{10} \text{N} + n.$

11. $\text{Log}_{10}\left(\frac{\text{N}}{10^n}\right) = \log_{10} \text{N} - n.$

12. $\text{Log}_a b \times \log_b a = 1.$

(9–12: See Todhunter's *Algebra for Colleges and Schools*, chap. xxviii.)

13. $\text{Log } (n + d) - \log n = \frac{\mu d}{n}$ (Todhunter's *Trigonometry*, Art. 177).

SIMPLE INTEREST.

P = principal in pounds,
n = number of years,
r = interest on £1 for 1 year,
M = amount,
R = amount of £1 for 1 year; $\therefore\ r = \text{R} - 1.$

Interest of £1 for n years $= nr$;

$\therefore$ interest of £P for n years $= \text{P}nr$;

14. $$\therefore\ \text{M} = \text{P} + \text{P}nr;$$

$$\therefore\ \text{P} = \frac{\text{M}}{1 + nr}.$$

COMPOUND INTEREST.

Amount of £1 for one year $= 1 + r = \text{R}$,
" " two years $= \text{R}^2$,
" " n years $= \text{R}^n$;
$\therefore$ amount of £P $= \text{PR}^n$;

15. $\therefore\ \text{M} = \text{PR}^n$, or $\log \text{M} = \log \text{P} + n \log \text{R}.$

16. Interest for n years $= \text{M} - \text{P} = \text{P}(\text{R}^n - 1).$

Interest payable half-yearly,

$$n \text{ becomes } 2n,\ r \text{ becomes } \frac{r}{2};$$

17. $$\therefore \text{ amount of P} = \text{P}\left(1 + \frac{r}{2}\right)^{2n}.$$

Similarly, if interest is payable q times a year,

18. $$\text{M} = \text{P}\left(1 + \frac{r}{q}\right)^{qn}.$$

Advantage of half-yearly payments.

19. Half-yearly $=\left(1+\frac{r}{2}\right)^2-(1+r)=\frac{r^2}{4}$ for one year.

Advantage in one year by q payments per annum

20. $$=\left(1+\frac{r}{q}\right)^{nq}-(1+r)^n.$$

A principal will become m times as much at simple interest when

$$mP=P+Pnr,$$
or when $m=1+nr$,
or when $n=\frac{m-1}{r}$;

21. that is, in $\frac{m-1}{r}$ years.

At compound interest, a principal will become m times as much when

$$mP=P(1+r)^n,$$
or when $m=(1+r)^n$,
or when $n=\frac{\log m}{\log(1+r)}$;

22. that is, in $\frac{\log m}{\log(1+r)}$ years.

DISCOUNT AT SIMPLE INTEREST.

$D=$ discount,
$V=$ present value.

23. $$V=\frac{P}{1+nr},$$

24. $$D=P-V=\frac{Pnr}{1+nr}.$$

DISCOUNT AT COMPOUND INTEREST.

25. $$V=\frac{P}{(1+r)^n},$$

26. $$D=P-\frac{P}{(1+r)^n}.$$

27. Whether interest is simple or compound,

$$\frac{1}{D}=\frac{1}{P}+\frac{1}{I},$$

for $V=\frac{P}{P+I}$; $\therefore D=\frac{PI}{P+I}$.

ANNUITIES.

P = present value of annuity,
A = amount of yearly payment,
M = amount of annuity in n years.

AT SIMPLE INTEREST.

28. $M = nA + \frac{n(n-1)}{2} rA;$

P amounts to $P + Pnr$ in n years;

$$\therefore\ P + Pnr = nA + \frac{n(n-1)}{2} rA;$$

29. $$\therefore\ P = \frac{nA + \frac{n(n-1)}{2} rA}{1 + nr}.$$

COMPOUND INTEREST.

Annuity continuing n years.

30. $M = \frac{R^n - 1}{R - 1} A.$ (Payments yearly.)

31. $$= \frac{\left(1 + \frac{r}{q}\right)^{nq} - 1}{\left(1 + \frac{r}{q}\right)^q - 1} A.$$ (Interest paid q times a year.)

32. $$= \frac{(1+r)^n - 1}{(1+r)^{\frac{1}{m}} - 1} \frac{A}{m}.$$ (Annuity paid m times a year in sums of £$\frac{A}{m}$. Interest yearly.)

33. $P = A\frac{1 - (1+r)^{-n}}{r}.$

Annuity continuing for ever.

34. $P = \frac{A}{r}.$

Annuity commencing p years hence, and continuing q years.

35. $P = \frac{A}{R-1}\left(\frac{1}{R^p} - \frac{1}{R^{p+q}}\right).$

Annuity commencing in p years and continuing for ever.

36. $P=\frac{A}{rR^p}$.

II. ALGEBRA.

Signs in fractions.

37. $\frac{a-b}{a-c}=\frac{b-a}{c-a}=-\frac{b-a}{a-c}=-\frac{a-b}{c-a}$.

Factors.

38. $x^2-y^2=(x+y)(x-y)$.

39. $x^3-y^3=(x-y)(x^2+xy+y^2)$.

40. $x^3+y^3=(x+y)(x^2-xy+y^2)$.

41. $x^4+x^2y^2+y^4=(x^2+xy+y^2)(x^2-xy+y^2)$.

42. $x^4-x^2y^2+y^4=(x^2+xy\sqrt{3}+y^2)(x^2-xy\sqrt{3}+y^2)$.

43. $x^4+4y^4=(x^2-2xy+2y^2)(x^2+2xy+y^2)$.

44. $x^n-y^n=(x-y)(x^{n-1}+x^{n-2}y+x^{n-3}y^2+$, etc.), n being any whole number.

45. $x^n-y^n=(x+y)(x^{n-1}-x^{n-2}y+x^{n-3}y^2-x^{n-4}y^3+$, etc.), n being even.

46. $x^n+y^n=(x+y)(x^{n-1}-x^{n-2}y+x^{n-3}y^2-x^{n-4}y^3+$, etc.), n being odd.

47. $x^3+y^3-z^3+3xyz=(x+y-z)(x^2+y^2+z^2+yz+xy+xz)$.

48. $x^3+y^3+z^3-3xyz=(x+y+z)(x^2+y^2+z^2-xy-xz-yz)$.

49. $x^3-y^3-z^3-3xyz=(x-y-z)(x^2+y^2+z^2+xy+xz-yz)$.

50. Ratio, $a^n:(a\pm x)^n::a:a\pm nx$ nearly.

51. $\frac{a}{b}=\frac{c}{a}=\frac{e}{f}=\left(\frac{pa^n+qc^n+re^n}{pb^n+qd^n+rf^n}\right)^{\frac{1}{n}}$.

For formulæ in proportion and variation, see pp. 35-37.

PERMUTATIONS AND COMBINATIONS.

Combinations of n things taken x at a time.

52. $=\frac{n(n-1)(n-2)\ .\ .\ .\ n-r+1}{\lfloor r}$

53. $=\frac{\lfloor n}{\lfloor r\lfloor n-r}$.

54. Permutations taken r at a time

$$=n(n-1)(n-2)\ldots(n-r+1).$$

55. Taken altogether $=1, 2, 3, \ldots = \underline{|n}$.

Permutations of n letters, p. A's, q. B's, r. C's, and all the rest unlike.

56. $$=\frac{\underline{|n}}{\underline{|p}\,\underline{|q}\,\underline{|r}}.$$

Combinations of two sets p and q, m being taken out of one set, n out of the other,

$$=\frac{p(p-1)(p-2)\ldots(p-m+1)}{1, 2, 3, \ldots m}$$
$$\times\frac{q(q-1)(q-2)\ldots(q-n+1)}{1, 2, 3, \ldots n}.$$

ARITHMETICAL PROGRESSION.

a = first term,
l = last term,
n = number of terms,
d = common difference,
x = a mean between two terms,
m = number of means.

57. $l=a+(n-1)d$.

58. $S=(2a+\overline{n-1}d)\,\frac{n}{2}$.

Thus, $S=a+(a+d)+(a+2d)+$, etc. $a+\overline{n-1}d$.

$$S=(a+\overline{n-1}d)+(a+\overline{n-2}d)+(a+\overline{n-3}d)$$
$$+\text{ etc.} \ldots a.$$

Adding $2S=(2a+\overline{n-1}d)+(2a+\overline{n-1}d)+(2a+\overline{n-1}d)$

$$+\text{ etc.} + \overline{2a+n-1d}.$$
$$=n(2a+\overline{n-1}d);$$
$$\therefore\ S=2a+\overline{n-1}d)\frac{n}{2}.$$

59. $S=(a+l)\frac{n}{2}$. See p. 42.

60. $d=\frac{n-1}{l-a}$. ,,

61. $n=\frac{2S}{a-l}$,,

62. $x=\frac{a+l}{2}$. See p. 42.

63. $d=\frac{l-a}{m+1}$. "

64. $d=\frac{2(S-an)}{n(n-1)}$.

65. $S=Mn$. M = middle one of an odd number of terms.

66. $S=\frac{(M'+M'')n}{2}$. M' M'' being the two middle terms of an even number of terms.

GEOMETRICAL PROGRESSION.

a = first term,
l = last term,
r = common ratio,
S = sum,
n = number of terms,
S = sum of n terms,
Σ = sum to infinity,
m = number of means,
x = one mean between two terms,

$$S=a+ar+ar^2+\text{ etc. } \dots ar^{n-1},$$
$$rS=ar+ar^2+\text{ etc. } \dots ar^{n-1}+ar^n.$$

Subtracting, $S(r-1)=ar^n-a$;

67. $\therefore S=\frac{a(r^n-1)}{r-1}=\frac{a(1-r^n)}{1-r}$.

68. $S=\frac{rl-a}{r-1}$.

69. $\Sigma=\frac{a}{1-r}$.

70. $r=\sqrt[n-1]{\frac{l}{a}}$.

71. $r=\sqrt[m+1]{\frac{l}{a}}$.

72. $\Sigma-S=\frac{ar^n}{1-r}$.

73. $x=\sqrt{ab}$.

74. $r=\frac{S-a}{S-l}$. See p. 53.

75. $1+2x+3x^2+4x^3+\text{etc. } \textit{ad infinitum} = \dfrac{1}{(1-x)^2}$.

76. „ „ to n terms $= \dfrac{nx^{n+1}-(n+1)x^n+1}{(1-x)^2}$.

See p. 54.

EQUATIONS, ETC.

77. $(a+x)^{-n} = \dfrac{1}{(a+x)^n}$.

78. $\dfrac{1}{(a+x)^{-n}} = (a+x)^n$.

79. $x = \dfrac{-b \mp \sqrt{4ac+b^2}}{2a}$. (Values of x in the equation $ax^2+bx+c=0$.)

80. Sum of roots $= -\dfrac{b}{a}$.

81. Product of roots $= \dfrac{c}{a}$.

82. Difference of roots $= \dfrac{\sqrt{b^2-4ac}}{a}$.

III. GEOMETRY.

MENSURATION.

Δ = area;
Sol. = solidity, volume, capacity;
a, b, c = sides of a triangle;
A, B, C = angles respectively opposite to a, b, c;
$S = \dfrac{a+b+c}{2}$.

LINES.

Right-angled triangle.

83. $h^2 = p^2 + b^2$.
h = height, p perpendicular, b = base.

Altitude of any triangle.

84. $p = \sqrt{\left\{a^2 - \dfrac{a^2+b^2-c^2}{4b^2}\right\}}$

$= \dfrac{1}{2b}\sqrt{\{(a+b+c)(a+b-c)(a-b+c)(b+c-a)\}}$.

b = base of triangle; a, c, sides of triangle; p = altitude.

Radius of circle inscribed in triangle. Base b, altitude p, and sum of three sides $a+b+c$.

85. Given radius $= \dfrac{pb}{a+b+c}$.

Circumference and diameter of circle.

86. $C = \pi D$.

$$D = \frac{C}{\pi}.$$

$$\pi = 3{\cdot}141592 65, \quad \frac{1}{\pi} = {\cdot}318309886.$$

SURFACES.

Area of rectangle.

87. $\Delta = ab$; a, b being adjacent sides.

Square.

88. $\Delta = a^2$

89. $\quad = \frac{1}{2}d^2$. $\qquad d =$ diagonal.

Parallelogram.

90. $b \times$ altitude. $\qquad b =$ base.

Trapezoid.

91. $\Delta = \dfrac{(S+S')B}{2}$.

S, S′ = parallel sides, B = base.

Triangle.

92. $\Delta = \dfrac{b \times \text{altitude}}{2}$. $\qquad b =$ base.

93. $\Delta = \sqrt{S(S-a)\,(S-B)\,(S-c)}$.

Any regular polygon.

94. $\Delta = \frac{1}{2}LNR$.

L = length of side, N = number of sides, R = radius of inscribed circle.

Circle.

95. $\Delta = \dfrac{C \times R}{2}$.

96. $\Delta = D^2 \times \dfrac{\pi}{4}$.

97. $\Delta = \dfrac{C^2}{4\pi}$.

C = circumference, D = diameter, R = radius.

Sector of circle.

$$\Delta = \frac{\text{Arc} \times R}{2}.$$

Circular ring.

98. $\Delta = (R^2 - r^2)\pi$.

99. $\Delta = \frac{1}{2}(C + c)(R - r)$.

C = outer circumference, c = inner circumference, R = outer radius, r = inner radius.

Convex surfaces. Right cylinder. Sides of right prism.

100. $\Delta = C \times L$.

C = circumference of base, L = length of prism or cylinder.

Right cone. Sides of right pyramid.

101. $\Delta = \frac{C \times S}{2}$.

C = circumference of base, S = slant height.

Sphere.

102. $\Delta = C \times D$.

103. $\Delta = D^2 \times \pi$.

104. $\Delta = C^2 \times \frac{1}{\pi}$.

105. $\Delta = 4R^2\pi$.

C = circumference, D = diameter.

SOLIDS.

Cube.

106. Sol. $= a^3$, a = edge.

Parallelopiped or cylinder.

107. Sol. $= A \times H$.

A = area of one end, H = height.

Pyramid or cone.

108. Sol. $= A \times \frac{H}{3}$.

Frustum of cone or pyramid.

109. Sol. $= \{E + e + \sqrt{Ee}\}\frac{H}{3}$.

Ee = areas of two ends.

Sphere.

110. Sol. $= D^3 \times \frac{\pi}{6}$.

TRIGONOMETRY.

111. Sin $(90 - A) = \cos A$.

112. Cos $(90 - A) = \sin A$.

113. Cot $(90 - A) = \tan A$.

114. Tan $(90 - A) = \cot A$.

115. Sin $(180 - A) = \sin A$.

160. $\dfrac{\text{Tan A}+\tan \text{B}}{\text{Tan A}-\tan \text{B}}=\dfrac{\sin (\text{A}+\text{B})}{\sin (\text{A}-\text{B})}$, by 127.

$$\left.\begin{array}{ll}
161. & \text{Sin } (A+B)+\sin (A-B)=2 \sin A \cos B \\
162. & \text{Sin } (A+B)-\sin (A-B)=2 \cos A \sin B \\
163. & \text{Cos } (A+B)+\cos (A-B)=2 \cos A \cos B \\
164. & \text{Cos } (A+B)-\cos (A-B)=2 \sin A \sin B
\end{array}\right\},$$

from 134, 135, 136, and 137, by addition and subtraction.

165. $\text{Sin A}+\sin \text{B}=2 \sin \frac{1}{2}(\text{A}+\text{B}) \cos \frac{1}{2}(\text{A}-\text{B})$.

166. $\text{Sin A}-\sin \text{B}=2 \cos \frac{1}{2}(\text{A}+\text{B}) \sin \frac{1}{2}(\text{A}-\text{B})$.

167. $\text{Cos A}+\cos \text{B}=2 \cos \frac{1}{2}(\text{A}+\text{B}) \cos \frac{1}{2}(\text{A}-\text{B})$.

168. $\text{Cos A}-\cos \text{B}=2 \sin \frac{1}{2}(\text{A}+\text{B}) \sin \frac{1}{2}(\text{A}-\text{B})$.

169. $\dfrac{\text{Cos B}-\cos \text{A}}{\text{Cos B}+\cos \text{A}}=\tan \frac{1}{2}(\text{A}+\text{B}) \tan \frac{1}{2}(\text{A}-\text{B})$.

170. $\dfrac{\text{Sin A}\pm\sin \text{B}}{\text{Cos A}+\cos \text{B}}=\tan \frac{1}{2}(\text{A}\pm\text{B})$.

171. $\dfrac{\text{Sin A}\pm\sin \text{B}}{\text{Cos B}-\cos \text{A}}=\text{cotan } \frac{1}{2}(\text{A}\mp\text{B})$.

172. $\text{Tan A}\pm\tan \text{B}=\dfrac{\sin (\text{A}\pm\text{B})}{\cos \text{A} \cos \text{B}}$.

173. $\text{Sin } (\text{A}+\text{B}) \sin (\text{A}-\text{B})=\sin^2 \text{A}-\sin^2 \text{B}$.

174. $\text{Cos } (\text{A}+\text{B}) \cos (\text{A}-\text{B})=\cos^2 \text{B}-\cos^2 \text{A}$.

175. $\text{Sin } n\text{A}+\sin (n-2) \text{A}=2 \sin (n-1) \text{A} \cos \text{A}$.

176. $\text{Cos } n\text{A}+\cos (n-2) \text{A}=2 \sin (n-1) \text{A} \cos \text{A}$.

177. $\dfrac{\text{Cos A}+\sin \text{A}}{\text{Cos A}-\sin \text{A}}=\sec 2\text{A}+\tan 2\text{A}$.

178. $\text{Cos } 2\text{A}=\dfrac{1}{1+\tan 2\text{A} \tan \text{A}}$.

179. $\text{Cos } 4\text{A}=2 \cos \text{A} \cos 3\text{A}-\sin 2\text{A}$.

180. $\text{Sin } 4\text{A}=2 \sin \text{A} \cos 3\text{A}-\sin 2\text{A}$.

181. $\text{Tan } 4\text{A}=\dfrac{4 \tan \text{A}-4 \tan^3 \text{A}}{1-6 \tan^2 \text{A}+\tan^4 \text{A}}$.

$$\left.\begin{array}{ll}
182. & \text{Sin } \frac{1}{2}A=\dfrac{\sqrt{1-\cos A}}{\sqrt{2}} \\
183. & \text{Cos } \frac{1}{2}A=\dfrac{\sqrt{1+\cos A}}{\sqrt{2}} \\
184. & \text{Tan } \frac{1}{2}A=\dfrac{\sqrt{1-\cos A}}{\sqrt{1+\cos A}}
\end{array}\right\} \text{from 142, 143.}$$

Triangles.

185. $\dfrac{\text{Sin A}}{a}=\dfrac{\sin B}{b}=\dfrac{\sin C}{c}$.

186. $\text{Cos A}=\dfrac{b^2+c^2-a^2}{2bc}$

187. $\text{Cos B}=\dfrac{a^2+c^2-b^2}{2ac}$ by Euclid II. 12, 13.

188. $\text{Cos C}=\dfrac{a^2+b^2-c^2}{2ab}$

189. $S=\frac{1}{2}(a+b+c)$.

190. $\text{Cos}\,\frac{1}{2}A=\sqrt{\dfrac{S(S-a)}{bc}}$.

191. $\text{Cos}\,\frac{1}{2}B=\sqrt{\dfrac{S(S-b)}{ac}}$, etc.

192. $\text{Sin}\,\frac{1}{2}A=\sqrt{\dfrac{(S-b)\,(S-c)}{bc}}$.

193. $\text{Sin}\,\frac{1}{2}B=\sqrt{\dfrac{(S-a)\,(S-c)}{ac}}$, etc.

194. $\text{Tan}\,\frac{1}{2}A=\sqrt{\dfrac{(S-c)\,(S-b)}{S(S-a)}}$.

195. $\text{Cot}\,\frac{1}{2}A=\sqrt{\dfrac{S(S-a)}{(S-b)\,(S-c)}}$.

$$\text{Sin A}=\frac{2}{bc}\sqrt{S(S-a)\,(S-b)\,(S-c)}.$$

Right-angled triangles. C = right angle.

c, A given.

196. $\dfrac{b}{c}=\cos A$.

197. $\dfrac{a}{c}=\sin A$.

198. $\text{Log}\,b=\log c+\log\cos A-10$.

199. $\text{Log}\,a=\log c+\log\sin A-10$.

A, b given.

200. $\dfrac{a}{b}=\tan A$.

201. $\text{Log}\,b=\log a-\log\tan A+10$.

202. $\dfrac{a}{c}=\sin A$, or $\log c=\log a-\log\sin A+10$.

a, b given.

203. $\text{Tan } A = \frac{a}{b}$; $\therefore \log \tan A = \log a - \log b + 10$.

204. $\frac{c}{b} = \sec A$; $\therefore \log c = \log b + \log \sec A - 10$.

c, a given.

205. $\text{Sin } A = \frac{a}{c}$; $\therefore \log \sin A = \log a - \log c + 10$.

206. $\frac{b}{c} = \cos A$; $\therefore \log b = \log c + \log \cos A - 10$.

207. $b^2 = c^2 - a^2$; $\therefore \log b = \frac{1}{2} \log (c+a) + \log (c-a)$.

OBLIQUE-ANGLED TRIANGLES.

A, C, b given.

208. $B = 180 - (A + C)$.

209. $a = b . \frac{\sin A}{\sin B}$; $\therefore \log a = \log b + \log \sin A - \log \sin b$.

210. $c = b . \frac{\sin C}{\sin B}$; $\therefore \log c = \log b + \log \sin C - \log \sin B$.

A, C, a given.

211. $B = 180 - (A + C)$.

212. $b = a . \frac{\sin B}{\sin A}$; $\therefore \log b = \log a + \log \sin B - \log \sin A$.

213. $c = a \frac{\sin C}{\sin A}$; $\therefore \log c = \log a + \log \sin C - \log \sin A$.

c, A, b given.

214. $B + C = 180 - A$.

215. $\text{Tan } \frac{1}{2}(B - C) = \frac{b-c}{b+c} \cot \frac{1}{2}A$;

$\therefore \log \tan \frac{1}{2}(B - C) = \log (b-c) - \log (b+c) + \log \cot \frac{1}{2}A$.

216. $B = \frac{1}{2}(B + C) + \frac{1}{2}(B - C)$.

217. $C = \frac{1}{2}(B + C) - \frac{1}{2}(B - C)$.

218. $a = c \frac{\sin A}{\sin C}$; $\therefore \log a = \log c + \log \sin A - \log \sin C$.

a, b, A given.

219. $\text{Sin } B = \frac{b}{a} \sin A$;

$\therefore \log \sin B = \log b + \log \sin A - \log a$.

220. $C = 180 - (A + B)$.

221. $c = a \frac{\sin C}{\sin A}$; $\therefore \log c = \log a + \log \sin C - \log \sin A$.

a, b, c given.

222. $\text{Sin } A = \frac{2}{bc}\sqrt{S(S-a)(S-b)(S-c)}$, or 190, 192.

AREA OF TRIANGLE.

223. $\Delta = \sqrt{S(S-a)(S-b)(S-c)}$.

224. $\Delta = \frac{1}{2}a^2 . \frac{\sin B . \sin C}{\sin (B+C)}$.

Circle in a triangle.

225. $\text{Rad.} = \frac{\sqrt{(S-a)(S-b)(S-c)}}{S}$.

Circle about a triangle.

226. $\text{Rad.} = \frac{abc}{4\sqrt{S(S-a)(S-b)(S-c)}}$.

227. Area of quadrilateral sides a, b, c, d.

$$\Delta = \sqrt{\{(S-a)(S-b)(S-c)(S-d)\}}.$$

USEFUL CONSTANTS.

$\sqrt{2} = 1 \cdot 4142$. $\sqrt{7} = 2 \cdot 6457$.

$\sqrt{3} = 1 \cdot 7320$. $\sqrt{10} = 3 \cdot 1622$.

$\sqrt{5} = 2 \cdot 2360$. $\sqrt{11} = 3 \cdot 3164$.

$\sqrt{6} = 2 \cdot 4494$. $\frac{1}{7} = \cdot\dot{1}4285\dot{7}$.

log 2 = ·3010300.	log 1·10 = ·0413927.
log 3 = ·4771213.	log 1·095 = ·0394141.
log 5 = ·6989700.	log 1·09 = ·0374265.
log 7 = ·8450980.	log 1·085 = ·0354297.
log 11 = 1·0413927.	log 1·08 = ·0334238.
log 13 = 1·1139434.	log 1·075 = ·0314085.
log 17 = 1·2304489.	log 1·07 = ·0293838.
log 19 = 1·2787536.	log 1·065 = ·0273496.
log 23 = 1·3617278.	log 1·06 = ·0253059.
log 29 = 1·4623980.	log 1·055 = ·0232525.
log 31 = 1·4913617.	log 1·05 = ·0211893.
log 37 = 1·5682017.	log 1·045 = ·0191163.
log 41 = 1·6127839.	log 1·04 = ·0170333.
log 43 = 1·6334685.	log 1·035 = ·0149403.

$\log 47 = 1{\cdot}6720979.$	$\log 1{\cdot}03 = {\cdot}0128372.$
$\log 53 = 1{\cdot}7242759.$	$\log 1{\cdot}025 = {\cdot}0107239.$
$\log 59 = 1{\cdot}7708520.$	$\log 1{\cdot}02 = {\cdot}0086002.$
$\log 61 = 1{\cdot}7853298.$	$\log 1{\cdot}015 = {\cdot}0064660.$
$\log 67 = 1{\cdot}8260748.$	$\log 1{\cdot}01 = {\cdot}0043214.$
$\log 71 = 1{\cdot}8582583.$	$\log 1{\cdot}005 = {\cdot}0021661.$
$\log 73 = 1{\cdot}8633229.$	
$\log 79 = 1{\cdot}8976271.$	
$\log 83 = 1{\cdot}9190781.$	
$\log 87 = 1{\cdot}9395193.$	
$\log 89 = 1{\cdot}9493900.$	
$\log 91 = 1{\cdot}9590414.$	
$\log 97 = 1{\cdot}9867717.$	

$1{\cdot}10^2 = 1{\cdot}21.$	$\log 1{\cdot}1^2 = {\cdot}0827854.$
$1{\cdot}10^3 = 1{\cdot}331.$	$\log 1{\cdot}1^3 = {\cdot}1241781.$
$1{\cdot}10^4 = 1{\cdot}4641.$	$\log 1{\cdot}1^4 = {\cdot}1655708.$
$1{\cdot}10^5 = 1{\cdot}61051.$	$\log 1{\cdot}1^5 = {\cdot}2069635.$
$1{\cdot}10^6 = 1{\cdot}771561.$	$\log 1{\cdot}1^6 = {\cdot}2483562.$

$1{\cdot}095^2 = 1{\cdot}199025.$	$\log 1{\cdot}095^2 = {\cdot}0788282.$
$1{\cdot}095^3 = 1{\cdot}312932.$	$\log 1{\cdot}095^3 = {\cdot}1182423.$
$1{\cdot}095^4 = 1{\cdot}437660.$	$\log 1{\cdot}095^4 = {\cdot}1576564.$
$1{\cdot}095^5 = 1{\cdot}574237.$	$\log 1{\cdot}095^5 = {\cdot}1970705.$
$1{\cdot}095^6 = 1{\cdot}723790.$	$\log 1{\cdot}095^6 = {\cdot}2364846.$

$1{\cdot}09^2 = 1{\cdot}1881.$	$\log 1{\cdot}09^2 = {\cdot}0743530.$
$1{\cdot}09^3 = 1{\cdot}295029.$	$\log 1{\cdot}09^3 = {\cdot}1122795.$
$1{\cdot}09^4 = 1{\cdot}411581.$	$\log 1{\cdot}09^4 = {\cdot}1497060.$
$1{\cdot}09^5 = 1{\cdot}538623.$	$\log 1{\cdot}09^5 = {\cdot}1871325.$
$1{\cdot}09^6 = 1{\cdot}677099.$	$\log 1{\cdot}09^6 = {\cdot}2245590.$

$1{\cdot}085^2 = 1{\cdot}177225.$	$\log 1{\cdot}085^2 = {\cdot}0708584.$
$1{\cdot}085^3 = 1{\cdot}277289.$	$\log 1{\cdot}085^3 = {\cdot}1062891.$
$1{\cdot}085^4 = 1{\cdot}387858.$	$\log 1{\cdot}085^4 = {\cdot}1417168.$
$1{\cdot}085^5 = 1{\cdot}505826.$	$\log 1{\cdot}085^5 = {\cdot}1771485.$
$1{\cdot}085^6 = 1{\cdot}633821.$	$\log 1{\cdot}085^6 = {\cdot}2125792.$

$1{\cdot}08^2 = 1{\cdot}1664.$	$\log 1{\cdot}08^2 = {\cdot}0668476.$
$1{\cdot}08^3 = 1{\cdot}259712.$	$\log 1{\cdot}08^3 = {\cdot}1002714.$
$1{\cdot}08^4 = 1{\cdot}360488.$	$\log 1{\cdot}08^4 = {\cdot}1336952.$
$1{\cdot}08^5 = 1{\cdot}469327.$	$\log 1{\cdot}08^5 = {\cdot}1671190.$
$1{\cdot}08^6 = 1{\cdot}586873.$	$\log 1{\cdot}08^6 = {\cdot}2005428.$

$1{\cdot}075^2 = 1{\cdot}155525.$	$\log 1{\cdot}075^2 = {\cdot}0628170.$
$1{\cdot}075^3 = 1{\cdot}242189.$	$\log 1{\cdot}075^3 = {\cdot}0942255.$
$1{\cdot}075^4 = 1{\cdot}335354.$	$\log 1{\cdot}075^4 = {\cdot}1256340.$
$1{\cdot}075^5 = 1{\cdot}435505.$	$\log 1{\cdot}075^5 = {\cdot}1570425.$
$1{\cdot}075^6 = 1{\cdot}543167.$	$\log 1{\cdot}075^6 = {\cdot}1884510.$
$1{\cdot}07^2 = 1{\cdot}1449.$	$\log 1{\cdot}07^2 = {\cdot}0587676.$
$1{\cdot}07^3 = 1{\cdot}225043.$	$\log 1{\cdot}07^3 = {\cdot}0881514.$
$1{\cdot}07^4 = 1{\cdot}310796.$	$\log 1{\cdot}07^4 = {\cdot}1175352.$
$1{\cdot}07^5 = 1{\cdot}402551.$	$\log 1{\cdot}07^5 = {\cdot}1469190.$
$1{\cdot}07^6 = 1{\cdot}500729.$	$\log 1{\cdot}07^6 = {\cdot}1763028.$
$1{\cdot}065^2 = 1{\cdot}134225.$	$\log 1{\cdot}065^2 = {\cdot}0546982.$
$1{\cdot}065^3 = 1{\cdot}207949.$	$\log 1{\cdot}065^3 = {\cdot}0820488.$
$1{\cdot}065^4 = 1{\cdot}286466.$	$\log 1{\cdot}065^4 = {\cdot}1093964.$
$1{\cdot}065^5 = 1{\cdot}370086.$	$\log 1{\cdot}065^5 = {\cdot}1367480.$
$1{\cdot}065^6 = 1{\cdot}459142.$	$\log 1{\cdot}065^6 = {\cdot}1640976.$
$1{\cdot}06^2 = 1{\cdot}1236.$	$\log 1{\cdot}06^2 = {\cdot}0506118.$
$1{\cdot}06^3 = 1{\cdot}191016.$	$\log 1{\cdot}06^3 = {\cdot}0759177.$
$1{\cdot}06^4 = 1{\cdot}262477.$	$\log 1{\cdot}06^4 = {\cdot}1012236.$
$1{\cdot}06^5 = 1{\cdot}338225.$	$\log 1{\cdot}06^5 = {\cdot}1265295.$
$1{\cdot}06^6 = 1{\cdot}418518.$	$\log 1{\cdot}06^6 = {\cdot}1518354.$
$1{\cdot}055^2 = 1{\cdot}113025.$	$\log 1{\cdot}055^2 = {\cdot}0465050.$
$1{\cdot}055^3 = 1{\cdot}174241.$	$\log 1{\cdot}055^3 = {\cdot}0697575.$
$1{\cdot}055^4 = 1{\cdot}238824.$	$\log 1{\cdot}055^4 = {\cdot}1030100.$
$1{\cdot}055^5 = 1{\cdot}306959.$	$\log 1{\cdot}055^5 = {\cdot}1162625.$
$1{\cdot}055^6 = 1{\cdot}378841.$	$\log 1{\cdot}055^6 = {\cdot}1395150.$
$1{\cdot}05^2 = 1{\cdot}1025.$	$\log 1{\cdot}05^2 = {\cdot}0423786.$
$1{\cdot}05^3 = 1{\cdot}157625.$	$\log 1{\cdot}05^3 = {\cdot}0635679.$
$1{\cdot}05^4 = 1{\cdot}21552625.$	$\log 1{\cdot}05^4 = {\cdot}0847572.$
$1{\cdot}05^5 = 1{\cdot}27631.$	$\log 1{\cdot}05^5 = {\cdot}1059465.$
$1{\cdot}05^6 = 1{\cdot}3401177.$	$\log 1{\cdot}05^6 = {\cdot}1271358.$
$1{\cdot}045^2 = 1{\cdot}09225.$	$\log 1{\cdot}045^2 = {\cdot}0382326.$
$1{\cdot}045^3 = 1{\cdot}141167.$	$\log 1{\cdot}045^3 = {\cdot}0573489.$
$1{\cdot}045^4 = 1{\cdot}192518+.$	$\log 1{\cdot}045^4 = {\cdot}0764652.$
$1{\cdot}045^5 = 1{\cdot}246181.$	$\log 1{\cdot}045^5 = {\cdot}0955815.$
$1{\cdot}045^6 = 1{\cdot}302260+.$	$\log 1{\cdot}045^6 = {\cdot}1146978.$
$1{\cdot}04^2 = 1{\cdot}0816.$	$\log 1{\cdot}04^2 = {\cdot}0340666.$
$1{\cdot}04^3 = 1{\cdot}124864.$	$\log 1{\cdot}04^3 = {\cdot}0510999.$

$1{\cdot}04^{4}=1{\cdot}169858.$	$\log 1{\cdot}04^{4}={\cdot}0681332.$
$1{\cdot}04^{5}=1{\cdot}216652.$	$\log 1{\cdot}04^{5}={\cdot}0851665.$
$1{\cdot}04^{6}=1{\cdot}265318.$	$\log 1{\cdot}04^{6}={\cdot}1021998.$
$1{\cdot}035^{2}=1{\cdot}071225.$	$\log 1{\cdot}035^{2}={\cdot}0298806.$
$1{\cdot}035^{3}=1{\cdot}108717.$	$\log 1{\cdot}035^{3}={\cdot}0448209.$
$1{\cdot}035^{4}=1{\cdot}147523.$	$\log 1{\cdot}035^{4}={\cdot}0597612.$
$1{\cdot}035^{5}=1{\cdot}187686.$	$\log 1{\cdot}035^{5}={\cdot}0747015.$
$1{\cdot}035^{6}=1{\cdot}229255.$	$\log 1{\cdot}035^{6}={\cdot}0896418.$
$1{\cdot}03^{2}=1{\cdot}0609.$	$\log 1{\cdot}03^{2}={\cdot}0256744.$
$1{\cdot}03^{3}=1{\cdot}092707.$	$\log 1{\cdot}03^{3}={\cdot}0385116.$
$1{\cdot}03^{4}=1{\cdot}125508.$	$\log 1{\cdot}03^{4}={\cdot}0513488.$
$1{\cdot}03^{5}=1{\cdot}159274.$	$\log 1{\cdot}03^{5}={\cdot}0641860.$
$1{\cdot}03^{6}=1{\cdot}194052.$	$\log 1{\cdot}03^{6}={\cdot}0770232.$
$1{\cdot}025^{2}=1{\cdot}050625.$	$\log 1{\cdot}025^{2}={\cdot}0214478.$
$1{\cdot}025^{3}=1{\cdot}076890.$	$\log 1{\cdot}025^{3}={\cdot}0321717.$
$1{\cdot}025^{4}=1{\cdot}03812.$	$\log 1{\cdot}025^{4}={\cdot}0428956.$
$1{\cdot}025^{5}=1{\cdot}131407.$	$\log 1{\cdot}025^{5}={\cdot}0536195.$
$1{\cdot}025^{6}=1{\cdot}159692.$	$\log 1{\cdot}025^{6}={\cdot}0643434.$
$1{\cdot}02^{2}=1{\cdot}0404.$	$\log 1{\cdot}02^{2}={\cdot}0172004.$
$1{\cdot}02^{3}=1{\cdot}061208.$	$\log 1{\cdot}02^{3}={\cdot}0258006.$
$1{\cdot}02^{4}=1{\cdot}08243.$	$\log 1{\cdot}02^{4}={\cdot}0344008.$
$1{\cdot}02^{5}=1{\cdot}104078.$	$\log 1{\cdot}02^{5}={\cdot}0432010.$
$1{\cdot}02^{6}=1{\cdot}126616.$	$\log 1{\cdot}02^{6}={\cdot}0516012.$
$1{\cdot}015^{2}=1{\cdot}030225.$	$\log 1{\cdot}015^{2}={\cdot}0129320.$
$1{\cdot}015^{3}=1{\cdot}045678.$	$\log 1{\cdot}015^{3}={\cdot}0193980.$
$1{\cdot}015^{4}=1{\cdot}061363.$	$\log 1{\cdot}015^{4}={\cdot}0258640.$
$1{\cdot}015^{5}=1{\cdot}077283.$	$\log 1{\cdot}015^{5}={\cdot}0323300.$
$1{\cdot}015^{6}=1{\cdot}093442.$	$\log 1{\cdot}015^{6}={\cdot}0387960.$
$1{\cdot}01^{2}=1{\cdot}0201.$	$\log 1{\cdot}01^{2}={\cdot}0086428.$
$1{\cdot}01^{3}=1{\cdot}030301.$	$\log 1{\cdot}01^{3}={\cdot}0129642.$
$1{\cdot}01^{4}=1{\cdot}040604.$	$\log 1{\cdot}01^{4}={\cdot}0172856.$
$1{\cdot}01^{5}=1{\cdot}051010.$	$\log 1{\cdot}01^{5}={\cdot}0216070.$
$1{\cdot}01^{6}=1{\cdot}061520.$	$\log 1{\cdot}01^{6}={\cdot}0259284.$
$1{\cdot}005^{2}=1{\cdot}05525.$	$\log 1{\cdot}01^{2}={\cdot}0043322.$
$1{\cdot}005^{3}=1{\cdot}060521.$	$\log 1{\cdot}01^{3}={\cdot}0064983.$
$1{\cdot}005^{4}=1{\cdot}065828.$	$\log 1{\cdot}01^{4}={\cdot}0086644.$
$1{\cdot}005^{5}=1{\cdot}071157.$	$\log 1{\cdot}01^{5}={\cdot}0108305.$
$1{\cdot}005^{6}=1{\cdot}076513.$	$\log 1{\cdot}01^{6}={\cdot}0129966.$

$$1+\frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2.3.4}+\text{ etc.}=e.$$

$e=2{\cdot}718281828$, the base of the Napierian logs.

Log 10 to base $e=2{\cdot}302581$.

$$\frac{1}{\log_e 10}=\frac{1}{2{\cdot}302581}={\cdot}434294481=\mu.$$

$\pi=3{\cdot}141592653.$ $\qquad \log\pi={\cdot}4971499.$

$\frac{\pi}{4}={\cdot}785398163.$ $\qquad \log\frac{\pi}{4}=\bar{1}{\cdot}8950899.$

$\frac{1}{\pi}={\cdot}318309886.$ $\qquad \log\frac{1}{\pi}=\bar{1}{\cdot}5028501.$

$\pi^2=9{\cdot}869604390.$ $\qquad \log\pi^2={\cdot}9942997.$

RADII OF CIRCLES INSCRIBED IN POLYGONS, SIDE BEING UNITY.

Trigon, . .	·28868.
Square, . .	·50000.
Pentagon, . .	·68819.
Hexagon, . .	·86603.
Heptagon, . .	1·03831.
Octagon, . .	1·20712.
Nonagon, . .	1·37373.
Decagon, . .	1·53881.
Undecagon, . .	1·70285.
Duodecagon, . .	1·86601.

TRIGONOMETRICAL FUNCTIONS.

3°.

$$\text{Sin } 3^\circ=\frac{1}{16}\{\sqrt{2}(\sqrt{15}-\sqrt{3}+\sqrt{5}-1)$$
$$-2(\sqrt{3}-1)\sqrt{(5+\sqrt{5})}\}={\cdot}052336.$$

$$\text{Cos } 3^\circ=\frac{1}{16}\{\sqrt{2}(\sqrt{15}-\sqrt{3}+\sqrt{5}+1)$$
$$+2(\sqrt{3}+1)\sqrt{5+\sqrt{5}}\}={\cdot}998630.$$

Log sin $3^\circ=8{\cdot}718800$, log cos $3^\circ=9{\cdot}999404$,
log tan $3^\circ=8{\cdot}719396$.

6°.

$$\text{Sin } 6^\circ=\frac{1}{8}(\sqrt{3}\sqrt{10-2\sqrt{5}})-(\sqrt{5}+1)={\cdot}104528.$$

$$\text{Cos } 6^\circ = \frac{1}{8}(\sqrt{10-2\sqrt{5}}) + \sqrt{3}(\sqrt{5}+1) = \cdot 994522.$$

Log sin $6^\circ = 9\cdot019235$, log cos $6^\circ = 9\cdot997614$,
log tan $6^\circ = 9\cdot021620$.

9°.

$$\text{Sin } 9^\circ = \frac{\sqrt{2}}{8}\{(\sqrt{5}+1) - \sqrt{10-2\sqrt{5}}\} = \cdot 156434.$$

$$\text{Cos } 9^\circ = \frac{\sqrt{2}}{8}\{(\sqrt{5}+1) + \sqrt{10-2\sqrt{5}}\} = \cdot 987688.$$

Log sin $9^\circ = 9\cdot194332$, log cos $9^\circ = 9\cdot994620$,
log tan $9^\circ = 9\cdot199713$.

12°.

$$\text{Sin } 12^\circ = \frac{1}{8}\{\sqrt{(10+2\sqrt{5})} - \sqrt{3}(\sqrt{5}-1)\} = \cdot 207912.$$

$$\text{Cos } 12^\circ = \frac{1}{8}\{\sqrt{3}\sqrt{10+2\sqrt{5}} + (\sqrt{5}-1)\} = \cdot 978148.$$

Log sin $12^\circ = 9\cdot317879$, log cos $12^\circ = 9\cdot990404$,
log tan $12^\circ = 9\cdot327475$.

15°.

$$\text{Sin } 15^\circ = \frac{1}{4}(\sqrt{6} - \sqrt{2}) = \cdot 258819.$$

$$\text{Cos } 15^\circ = \frac{1}{4}(\sqrt{6} + \sqrt{2}) = \cdot 965926.$$

Log sin $15^\circ = 9\cdot412996$, log cos $15^\circ = 9\cdot984944$,
log tan $15^\circ = 9\cdot428052$.

18°. $$\text{Sin } 18^\circ = \frac{1}{4}(\sqrt{5} - 1) = \cdot 309017.$$

$$\text{Cos } 18^\circ = \frac{1}{4}\{\sqrt{10+2\sqrt{5}}\} = \cdot 951057.$$

Log sin $18^\circ = 9\cdot489982$, log cos $18^\circ = 9\cdot978206$,
log tan $18^\circ = 9\cdot511776$.

21°.

$$\text{Sin } 21^\circ = \frac{\sqrt{2}}{16}\{\sqrt{5}+1+\sqrt{10-2\sqrt{5}} - \sqrt{3}\,(\sqrt{5}+1) + \sqrt{3}\sqrt{10-2\sqrt{5}}\} = \cdot 358368.$$

$$\text{Cos } 21^\circ = \frac{\sqrt{2}}{16}\{\sqrt{3}(\sqrt{5}+1) + \sqrt{3}\sqrt{10-2\sqrt{5}} + \sqrt{5} + 1 - \sqrt{10-2\sqrt{5}}\} = \cdot 933580.$$

$$\text{Log}\sin 21^\circ = 9{\cdot}554329,\ \log\cos 21^\circ = 9{\cdot}970152,$$
$$\log\tan 21^\circ = 9{\cdot}584177.$$

$22\frac{1}{2}^\circ$.

$$\text{Sin } 22\tfrac{1}{2}^\circ = \frac{1}{2}\{\sqrt{2-\sqrt{2}}\} = {\cdot}382683.$$

$$\text{Cos } 22\tfrac{1}{2}^\circ = \frac{1}{2}\{\sqrt{2+\sqrt{2}}\} = {\cdot}923880.$$

$$\text{Log}\sin 22\tfrac{1}{2}^\circ = 9{\cdot}582840,\ \log\cos 22\tfrac{1}{2}^\circ = 9{\cdot}965615,$$
$$\log\tan 22\tfrac{1}{2}^\circ = 9{\cdot}617224.$$

24°.

$$\text{Sin } 24^\circ = \frac{1}{16}\{\sqrt{3}\sqrt{10-2\sqrt{5}} + \sqrt{5} + 1 - 3\sqrt{10-2\sqrt{5}}$$
$$+\sqrt{3}(\sqrt{5}+1))\} = {\cdot}406737.$$

$$\text{Cos } 24^\circ = \frac{1}{16}\{3\sqrt{10-2\sqrt{5}} + \sqrt{3}(\sqrt{5}+1)$$
$$+\sqrt{3}\sqrt{10-2\sqrt{5}} - \sqrt{5} - 1\} = {\cdot}913545.$$

$$\text{Log}\sin 24^\circ = 9{\cdot}609313,\ \log\cos 24^\circ = 9{\cdot}960730,$$
$$\log\tan 24^\circ = 9{\cdot}648583.$$

30°.

$$\text{Sin } 30^\circ = \frac{1}{2} = {\cdot}50000.$$

$$\text{Cos } 30^\circ = \frac{\sqrt{3}}{2} = {\cdot}848048.$$

$$\text{Log}\sin 30^\circ = 9{\cdot}698970,\ \log\cos 30^\circ = 9{\cdot}937531,$$
$$\log\tan 30^\circ = 9{\cdot}761439.$$

36°.

$$\text{Sin } 36^\circ = \frac{1}{4}\{\sqrt{10-2\sqrt{5}}\} = {\cdot}587785.$$

$$\text{Cos } 36^\circ = \frac{1}{4}\{1\sqrt{5}+1\} = {\cdot}788011.$$

$$\text{Log}\sin 36^\circ = 9{\cdot}769219,\ \log\cos 36^\circ = 9{\cdot}907958,$$
$$\log\tan 36^\circ = 9{\cdot}861261.$$

45°.

$$\text{Sin } 45^\circ = \frac{1}{\sqrt{2}} = {\cdot}7071.$$

$$\text{Cos } 45^\circ = \frac{1}{\sqrt{2}} = {\cdot}7071.$$

$$\text{Log sin } 45^\circ = 9{\cdot}849485, \text{ log cos } 45^\circ = 9{\cdot}849485,$$
$$\text{log tan } 45^\circ = {\cdot}00000.$$

The sines of 48°, 51°, 54°, 57°, 60°, 63°, 66°, 67½°, 69°, 72°, 75°, 78°, 81°, 84°, 87°, will be the same as the cosines of these angles subtracted from 90°.

The cosines of these values will be the same as the sines of these values subtracted from 90°.

Log tangents for values above 45° will be the same as the log cotangents of these values subtracted from 90°.

The cosines, sines, and tangents of 93°, 96°, etc., may be found from these values by remembering that

$$\sin A = \sin (180 - A),$$
$$\cos A = -\cos (180 - A).$$

The functions of angles between 180 and 270 may also be calculated from them, and

$$\sin A = -\sin (180 + A),$$
$$\cos A = -\cos (180 + A).$$

EXERCISES.

Of the following questions, those numbered 1 to 30, 51 to 78, 101 to 125, 131 to 141, 151 to 163, 171 to 178, 181 to 190, 201 to 211, 231 to 245, 301 to 390, 401 to 450, are the questions given at the 1st B.A. and 1st B.Sc. Examinations at London University in the years 1867 to 1878.

EXERCISES.

SECTION I.—ARITHMETIC.

ORDINARY RULES OF ARITHMETIC.

1. If the length of the year is 365·242264 days, but is reckoned as $365\frac{1}{4}$ days, find in how many centuries the accumulated error would amount to $263\frac{3}{125}$ days.

2. Find the G. C. M. of 11310 and 86478, of 86478 and 448630, and of 11310, 86488, and 448630.

3. Find the prime factors of 6930, 1470, and 5775, and use them for calculating (1) the sum of the reciprocals, and (2) the square root of the product of the three numbers.

4. State and prove the rule for finding the G. C. M. of two numbers.

5. A person, having asked the time of day, is told that it is between five and six o'clock, and the hour and minute hands are together. What o'clock is it?

6. Find the income arising from investing £740 in the Three per Cents. at $92\frac{1}{2}$.

7. Find the greatest divisor of 1287000 and 504504, and prove that every common divisor of two given numbers divides their greatest common divisor.

8. How long will an up train and a down train be in passing one another, if each of them be 44 yards long, and if each of them travels at the rate of 30 miles an hour?

9. A metre being 39·37 inches, state accurately, as far as three places of decimals, what decimal fraction a foot is of a metre.

10. Reduce the fraction $\frac{17427}{24975}$ to its lowest terms.

11. If 640 acres go to a square mile, what is the length of the sides of a square plot of ground which contains 100 acres?

12. Assuming that 1 rood = 40 square perches, 1 perch = $30\frac{1}{4}$ square yards, calculate to the nearest integer the

length in feet of each side of a square courtyard of 1 rood in extent.

13. Divide £26, 3s. 3d. between three persons, so that their shares may be to one another in the proportion of the fractions $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$.

14. A person invested in the Three per Cents. at $94\frac{1}{2}$, and received interest £200 a year. What sum did he invest?

VULGAR AND DECIMAL FRACTIONS.

15. State and prove the rule for the division of decimals.

16. Simplify $\frac{1\cdot18}{\cdot152} \times \frac{3\cdot04}{2\cdot95}$, and divide the result by $\cdot00125$.

17. Find the value of $\frac{\cdot011 \times 133\cdot1 - \cdot723 \times \cdot00723}{1\cdot1377}$, and express as a fraction of 9 seconds $\cdot00002578125$ of $3\frac{1}{3}$ days.

18. Find the vulgar fraction equivalent to $\cdot0\dot{7}14828\dot{5}$, and the circulating decimal equivalent to $\frac{1}{\cdot1001}$

19. If n be a whole number, state in what cases the decimal equivalent to $\frac{1}{n}$ terminates, and in what case it circulates.

20. Convert the circulating decimal $1\cdot\dot{4}63\dot{1}$ to a vulgar fraction.

21. Find the value of

$$\frac{\cdot13 \times \cdot14 \times \cdot01 - \cdot12 \times \cdot14 \times \cdot02 + \cdot12 \times \cdot13 \times \cdot01}{\cdot01 \times \cdot2 \times \cdot01}.$$

22. Reduce 2 days 9 hours to the decimal of a week.

23. Find the value of $\frac{\cdot05 \times \cdot05 \times \cdot05 + 1}{1\cdot05}$, and of $\cdot\dot{4}2857\dot{1}$ of 1 minute 7 seconds.

THE RULE OF THREE AND ITS APPLICATIONS.

24. Find to the nearest shilling the present value of £273, payable after three years, at 3 per cent. per annum interest.

25. The population of a country is 32,000,000, and it increases at the rate of 5 per cent. every year. What will it be at the end of 5 years?

26. What fraction is that from which if $\frac{5\frac{1}{7}}{15\frac{3}{7}}$ be subtracted, and the remainder be divided by $\frac{7\frac{1}{8}}{32\frac{1}{16}}$, the result will be $\frac{1}{12}$?

27. Extract the square root of 10 to four places of decimals. How many more places can you get by simple division? By aid of the value of $\sqrt{10}$ thus obtained, find $\sqrt{\cdot 004}$.

28. Extract the square root of 491401; and reduce 2 days 9 hours to the decimal of a week.

29. Find the square root of the circulating decimal ·111. . . . State any law that you may notice in the form of the digits which express the successive remainders.

30. Calculate to the nearest halfpenny the true present value of a bill for £152, 8s., due December 31, and discounted on July 17 at $3\frac{1}{2}$ per cent.

31. Find the value of 6413 things at 4s. $10\frac{7}{16}$d. each.

32. A man who owns $\frac{4}{17}$ of an estate, sold $\frac{2}{33}$ of his share for $\frac{1}{88}$ of £400. What was the value of $\frac{2}{17}$ of the whole estate?

33. A gentleman leaves £3000 to be divided between three companions in the proportion of 1, 2, 3. Find the amount each will receive after paying a legacy duty of 10 per cent.

34. The difference between the simple and compound interest on a sum of money at 5 per cent. for 2 years is £2·555. What is the principal?

35. At what time between three and four o'clock are the hands of a clock at an angle of 15°?

36. Reduce

$$16\left(\frac{1}{5}-\frac{1}{3\times5^3}+\frac{1}{5\times5^5}-\frac{1}{7\times5^7}+\frac{1}{9\times5^9}-\text{etc.}\right)-\frac{4}{239}$$

to a decimal to four places.

37. Simplify $\frac{\cdot375\times\cdot375-\cdot025\times\cdot025}{\cdot375-\cdot025}$.

38. If A and B can do a piece of work in 12 days, B and C in 14 days, and A, B, and C in 8 days, how long will it take A and C to do it when they work together?

39. Find the value of $\frac{11}{133}$ of £300, and express $\frac{5\frac{1}{2}}{66\frac{1}{2}}$ as a decimal.

40. A woman bought an equal number of oranges at 2 for a penny and 3 for a penny, and by mixing them and selling them at 5 for 2d., she lost 10d. How many did she buy?

41. Calculate the value of $1+\frac{1}{1}+\frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2.3.4}$ + etc., to 8 places of decimals.

42. What is the interest on £1578 for 139 days at $4\frac{1}{4}$ per cent. per annum? and on £457, 8s. for 7 months at the same rate?

43. A garrison of 1354 soldiers are found to have consumed 12,190 stones of salt beef in 127 days. After that they receive a reinforcement of 352 men. What weight of the same article must be laid in to maintain the whole number 119 days longer?

44. An estate of 386 imperial acres was purchased for £19,740. A small piece of land lies contiguous, which contains 3·5 Scotch acres of equal quality. The owner offers to sell this piece on the same terms as the estate. What will be the price, the Scotch acre being equivalent to 1·261 imperial acres?

45. If the gilding of a wall $6\cdot\dot{4}2857\dot{1}$ yards long, and 1·78 yards wide, cost 25s. a square inch, what is the total cost?

46. Reduce $\frac{2^7}{7\times 10^7}+\frac{2^3}{3\times 10^3}$ to a decimal.

47. Extract the square root of 53111·8116, of 531·118116, of 5·31118116, and of ·0531118116, by the shortest methods.

48. Find the square roots of $\frac{529}{2401}$, of $\frac{5}{7}$, and of $\frac{5\cdot 04}{\cdot 021}$.

49. Simplify $\left(\frac{2\frac{1}{4}-\frac{2}{3}\text{ of }1\frac{5}{6}}{\frac{1}{5}\text{ of }3\frac{1}{3}+\frac{13}{36}}-\frac{1}{2\frac{1}{2}}\right)$ divided by $\frac{1}{1\frac{2}{3}}$.

50. At what time between one and two o'clock are the hour and minute hands, (1) together, (2) in a straight line in opposite directions?

LOGARITHMS: COMPOUND INTEREST AND ANNUITIES.

51. Find the amount of an annuity left unpaid for any number of years, allowing compound interest.

52. A person starts with a certain capital, which produces him 4 per cent. per annum compound interest. He spends every year a sum equal to twice the original interest on his capital. Find in how many

years he will be ruined, having given log $2 = \cdot 3010300$, log $13 = 1\cdot 1139434$.

53. Investigate the rule to find the discount on any sum, £A, due t years hence, at r per cent. per annum.

54. If log $4 = \cdot 6020600$, log $27 = 1\cdot 4313638$, log $7 = \cdot 8450980$, find log $\cdot 0027$ and log 3528.

55. Find the present value of an annuity of £A, to continue for n years, allowing compound interest at the rate of r per cent. per annum.

56. Define a logarithm, and prove that the logarithm of a product of two factors is equal to the sum of the logs of the two factors.

57. Given log $2 = \cdot 30103$, what are the logs of $2^{-\frac{1}{5}}$, of $\cdot 00002$, of $62\cdot 5$, and $5^{-\frac{1}{4}}$?

58. Find the value of a perpetual annuity of £225 per annum at $3\frac{1}{2}$ per cent. rate of interest.

59. Find the amount of an annuity of £1 per annum in n years at r per cent. interest.

60. What is the characteristic of the logarithm of 2000 to the base 3? If the mantissæ of the logs of 9450, 9451, to the base 10 are 9754318 and 9754778 respectively, find the complete logarithm to the same base of 9450666 by the method of proportional parts.

61. An annuity of £P per annum is to begin n years hence, and is to be payable for ever. Find its present value at r per cent., and show that its present value is to its value n years hence as $(1+r)^{-n} : 1$.

62. What is meant by the base of a system of logarithms? What are the advantages of taking 10 as the base?

63. Prove that if $N = \frac{P}{Q}$, log $N = \log P - \log Q$, and find as far as four places of decimals the number of which the logarithm to base 10 is $\cdot 5$.

64. If an annuity continued for ever is worth 25 years' purchase, what annuity, at the same rate of interest, to be continued for 3 years, can be purchased for £5000?

65. What is the characteristic of the logarithm of 50 to the base $\sqrt{2}$? If the log of 3 to the base 10 is $\cdot 477121$, what is its log to the base $\sqrt[5]{10}$?

66. Given log $648 = 2\cdot 81157501$, log $864 = 2\cdot 93651374$, find the log of 108.

67. What annuity, beginning n years hence, and

lasting n years, is equal in value to an annuity of £A beginning now and lasting for n years, interest being reckoned at R − 1 per cent.

68. Given log $2 = \cdot301030$, log $3 = \cdot477121$, find the logs of $\cdot00625$, $\frac{1}{24}$, and $(\cdot0003)^5$. Find also approximately the value of x which satisfies the equation $2^x = 5$.

69. What is the present value of an annuity of a given amount per annum in perpetuity, when the interest is r per cent. If such an annuity is worth 25 years' purchase, what is the value of an annuity of £1 at the end of the first year, £2 at the end of the second year, £3 at the end of the third year, and so on, continued for ever.

70. The mantissæ of the logs of 2, 3, 11 being ·301030, ·477121, ·041393, find in how many years the population of a country will first become increased by more than one-half of its original amount, through the sole effect of births and deaths, when the birth-rate at the end of each year is $\frac{1}{30}$ and the death-rate $\frac{1}{40}$ of the population at the beginning of that year.

71. Find the present value of an annuity of £20 for five years at $3\frac{1}{2}$ per cent., to commence at the end of 20 years. What would be the value if a half-yearly payment of £10 were substituted for a yearly payment of £20?

72. What is meant by the logarithm of a number? and what by a system of logarithms?

Having given a system of logs to the base a, how may the logarithms to a base b be calculated?

73. Having given
$$\log 3796 = 3\cdot5793262,$$
$$\log 2984 = 3\cdot4747988,$$
$$\log 90714 = 4\cdot9576743,$$
$$\log 90715 = 4\cdot9576791,$$
calculate to seven places the value of
$$\sqrt[5]{\frac{(\cdot3796)^3}{(\cdot2984)^2}}.$$

74. State and prove the ordinary formula for the present value, at n per cent. compound interest, of a deferred annuity, to commence at the end ot p years, and to continue for q years.

75. If x be a large positive number, and $\pm\delta$ a comparatively small increase or diminution of it; prove the

approximate formulæ $\log(x \pm \delta) = \log x \pm \mu\frac{\delta}{x}$, where μ = modulus of the system employed.

76. Having given $\log 193{\cdot}06 = 2{\cdot}2856923$,
$\log 19307 = 4{\cdot}2857148$,
find the seventh root of 100 to six decimals.

77. Assuming the fundamental properties of logarithms, state and prove the ordinary rules by which the logarithm of a number not given in the tables, and the number corresponding to a logarithm not given in the tables may be calculated from the tables.

78. Given $\log 2 = {\cdot}30102999561$, $\log 3 = {\cdot}4771212546$, calculate to ten places the logs of 4·5, 6·75, and 10·125 in the same system.

79. What is the amount of £10 for 50 years at 4 per cent. per annum compound interest?

80. Find the compound interest on £383 for $5\frac{1}{2}$ years at $3\frac{1}{2}$ per cent. per annum.

81. What sum will a penny amount to in 500 years at 5 per cent. per annum compound interest?

82. In what time will £1 amount to £10 at 5 per cent. interest?

83. What will a farthing amount to in 500 years at 3 per cent. per annum compound interest.

84. Having given log 2 and log 3, find the logs of 6·4, $(1\frac{1}{2})^{20}$, and 1·25.

85. What sum of money must I invest at 5 per cent. compound interest, so that in 12 years I may have £500? $\text{Log } 1{\cdot}05 = {\cdot}0219$.

86. If the interest is payable half-yearly, what will £500 amount to in 8 years at 5 per cent.?

87. Given that to five places of decimals, $\mu = {\cdot}43429$ in the ordinary system for which the base is 10; calculate to that number of places the logs of 999 and of 1001 in that system.

88. Having given $\log 112 = 2{\cdot}0492180$, $\log 196 = 2{\cdot}2922561$, $\log 441 = 2{\cdot}6444386$, find the logs of 2, 3, 7, and 42.

89. How many digits will there be in 2^{64}?

90. Having given $\log 2 = {\cdot}30103$, find log 199 and log 201.

91. Prove that $\log_a b \times \log_b a = 1$.

92. What is the present value of £600 due 3 years hence at 5 per cent. compound interest?

93. What will be the amount of £100 in 1 year at 5 per cent., if the interest is paid half-yearly.

94. Show that whether simple or compound interest is reckoned, the discount on any sum is always less than the interest.

95. Given that log $2 = \cdot 30103$, find the log of 16^{20}.

96. Having given log $2 = \cdot 30103$, log $43 = 1\cdot 63346$, and log $3 = \cdot 477121$, find the log of $1\cdot 08^5 \times 2 + 1\cdot 08^4 \times 3$.

97. Having given log $448 = 2\cdot 651278$, log $392 = 2\cdot 593286$, find log 2, 5, 25, 125.

98. Having given log 2, log 3, and log $1\cdot 2597 = 100269$, find the compound interest on £600 for 3 years at 8 per cent. per annum.

99. Find the logarithm of $\sqrt[5]{7\sqrt{2} \times \sqrt[8]{3}}$, having given log 2, log 3, and log $7 = \cdot 845098$.

100. Having log 2 and log 7, find the log of $\left(\frac{5}{7}\right)^{\cdot 05347}$.

SECTION II.—ALGEBRA.

101. Extract the square root of

$$4x^4 - 12x^3y + 25x^2y^2 - 24xy^3 + 16y^4.$$

102. Simplify $\dfrac{x^2 - x + 1}{x^2 + x + 1} + \dfrac{2x(x-1)^2}{x^4 + x^2 + 1} + \dfrac{2x^2(x^2-1)^2}{x^8 + x^4 + 1}$.

103. Simplify $\dfrac{x^3 - 4x^2 + 5x - 2}{x^2 - 1}$, and find the factors of $x^3 + 3axy + y^3 - a^3$.

104. If $x = \sqrt[3]{-1 + \sqrt{2}} + \sqrt[3]{-1 - \sqrt{2}}$, find the value of $x^3 + 3x + 2$.

105. Simplify the expressions—

$$\frac{a^2}{(a-b)(a-c)} + \frac{b^2}{(b-c)(b-a)} + \frac{c^2}{(c-a)(c-b)},$$

$$\text{and } \frac{\sqrt{12}}{(1+\sqrt{2})(\sqrt{6}-\sqrt{3})}.$$

106. If $x^2 + 2ax - 3b^2$ is divisible by $x - a$ without a remainder, show that $a = +b$ or $-b$.

107. Simplify the expressions,

$$(\alpha)\ \frac{a+b}{a-b}+\frac{b+c}{b-c}+\frac{c+a}{c-a}+\frac{(a+b)(b+c)(a+c)}{(a-b)(b-c)(c-a)},$$

$$(\beta)\ \left(\frac{a}{b}+\frac{b}{a}\right)^2+\left(\frac{b}{c}+\frac{c}{b}\right)^2+\left(\frac{c}{a}+\frac{a}{c}\right)^2-\left(\frac{a}{b}+\frac{b}{a}\right)\left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{c}{a}+\frac{a}{c}\right).$$

108. Simplify $\dfrac{(ay-bx)^2+(ax+by)^2}{\left(\frac{a}{b}+\frac{b}{a}\right)\left(\frac{x}{y}+\frac{y}{x}\right)}$, and divide $1+10x^3+27x^6$ by $1-2x+3x^2$.

109. Simplify $\sqrt{\dfrac{147}{605}}\left(\dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}-\dfrac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}\right)$.

110. Express the fraction $\dfrac{1-\sqrt{2}+\sqrt{5}}{1+\sqrt{2}-\sqrt{5}}$ under the form of a fraction with a rational denominator.

111. Find the value of

$$\frac{(a^2+b^2-c^2)(a^2+c^2-b^2)+(b^2+c^2-a^2)(b^2+a^2-c^2)+(c^2+a^2-b^2)(c^2+b^2-a^2)}{(a+b+c)(a+c-b)(b+c-a)(a+b-c)}$$

and divide $\dfrac{x-a}{x+a}-\dfrac{x^3-a^3}{x^3+a^3}$ by $\dfrac{x+a}{x-a}+\dfrac{x^2+a^2}{x^2-a^2}$.

112. Extract the square root of

$$x^4+x^3+\frac{5}{4}x^2+\frac{5}{2}x+\frac{5}{4}+\frac{1}{x}+\frac{1}{x^2}.$$

113. Simplify—

(α) $(x-y)^3+(y-z)^3+(z-x)^3-3(x-y)(y-z)(z-x)$.

$$(\beta)\ x^3+\cfrac{x^2}{x^2+\cfrac{1}{x^2-\cfrac{x^8+x^3-1}{x^5}}}.$$

$$(\gamma)\ \frac{1}{a-2b}-\frac{2}{a-b}+\frac{2}{a+b}-\frac{1}{a+2b}.$$

114. Simplify the expression

$$\frac{\sqrt{a+b}+\sqrt{a-b}}{\sqrt{a+b}-\sqrt{a-b}}-\frac{\sqrt{a+b}-\sqrt{a-b}}{\sqrt{a+b}+\sqrt{a-b}}.$$

115. Divide $a^2-b^2-c^2-2bc$ by $\dfrac{a+b+c}{a+b-c}$.

116. Find the square of

$$\frac{\sqrt{2ab+(a^2+b^2)\sqrt{-1}}+\sqrt{2ab-(a^2+b^2)\sqrt{-1}}}{a+b}.$$

117. Simplify the expression

$$\left(\frac{y-z}{x}+\frac{z-x}{y}+\frac{x-y}{z}\right)\left(\frac{x}{y-z}+\frac{y}{z-x}+\frac{z}{x-y}\right)$$
$$+\left(\frac{z-x}{x-y}-\frac{x-y}{z-x}\right)\left(\frac{y}{z}-\frac{z}{y}\right)+\left(\frac{x-y}{y-z}-\frac{y-z}{x-y}\right)\left(\frac{z}{x}-\frac{x}{z}\right)$$
$$+\left(\frac{y-z}{z-x}-\frac{z-x}{y-z}\right)\left(\frac{x}{y}-\frac{y}{x}\right).$$

118. Prove that if the four fractions,

$$\frac{bx+cy+dz}{b+c+d-a},\ \frac{cx+dy+az}{c+d+a-b},\ \frac{dx+ay+bz}{d+a+b-c},\ \frac{ax+by+cz}{a+b+c-d}$$

are equal to one another, their common value will be equal to $\frac{x+y+z}{2}$ as long as $a+b+c+d$ does not vanish; but if $a+b+c+d=0$, the quantities x, y, z must be equal to one another, and then half their common value will be the common value of the fractions.

119. Simplify

$$\frac{x^2-yz}{(x-y)(x-z)}+\frac{y^2+xz}{(y+z)(y-x)}+\frac{z^2+xy}{(z-x)(z+y)}.$$

120. Verify, by multiplication, the algebraic identity,

$$(bcx+cay+abz-xyz)^2+(ayz+bxz+cxy-abc)^2$$
$$=(a^2+x^2)(b^2+y^2)(c^2+z^2).$$

121. Extract the square root of $1+2x+3x^2+4x^2+$ etc., to infinity by the ordinary process.

122. If $k=x\sqrt{1+y^2}+y\sqrt{1+x^2}$, prove that $\sqrt{1+k^2}$ $=xy+\sqrt{1+y^2}\sqrt{1+x^2}$.

123. Show that

$$(y-z)^3+(z-x)^3+(x-y)^3=3(y-z)(z-x)(x-y).$$

124. Find by multiplication or otherwise, and arrange in ascending powers of x, the square of

$$1+\frac{1}{2}x+\frac{13}{24}x^2+\frac{135x^3}{246}+\frac{1357}{2468}x^4+\text{etc.}$$

to infinity.

125. Extract the square root of $1+x+x^2+x^3+$ etc. to infinity, and arrange according to ascending powers of x.

126. Prove that $x^4+px^3+qx^2+rx+s$ is a complete square of $p^2s=r^2$, and $q=\frac{p^2}{4}+2\sqrt{s}$.

127. If $x=\frac{a+b}{c-d}$, show that $(a-cx)^2+(x^2-1)(b^2-d^2)$ is a complete square.

128. If $x=\frac{2ac}{b(1+c^2)}$, find the value of

$$\frac{\sqrt{a+bx}+\sqrt{a-bx}}{\sqrt{a+bx}-\sqrt{a-bx}}.$$

129. Prove that the sum of all numbers composed of the same digits is divisible by the sum of the digits.

130. If $\left(\frac{s}{3}\right)^3=\left(\frac{r}{4}\right)^4$, the expression x^4+rx+s has a factor of the form $(x+a)^2$. Prove this.

RATIO, PROPORTION, VARIATION.

131. If four numbers are proportional, the product of the extremes is equal to the product of the means.

132. If $a : b :: c : d$, find the relation between p, q, r, s, in order that $(pa+qb+rc+sd)\ (pa-qb-rc+sd)$ may be equal to $(pa-qb+rc-sd)\ (pa+qb-rc-sd)$.

133. If $a : b :: c : d$, prove that $a+b : b :: c+d : d$. What quantity must be added to each of the terms of the ratio $\frac{a}{b}$, so that it may be equal to the ratio $\frac{e}{f}$?

134. If $a : b :: b : c :: c : d$, prove that $a : d :: a^3 : b^3$.

135. A varies partly directly as B and partly inversely as B. If $A=3$ when $B=1$, and also when $B=2$, what will be the value of B when $A=4\frac{1}{2}$.

136. If $A \propto B$ when C is invariable, and $A \propto C$ when B is invariable, prove that $A \propto BC$ when both B and C are variable.

137. The total increase in the number of patients in a certain hospital this year over the number in the year preceding was $2\frac{1}{2}$ per cent. In the number of out-patients there was an increase of 4 per cent., but in that of the in-patients a decrease cf 11 per cent. Find the ratio of the number of out-door patients to the number of in-door patients.

138. If four numbers are proportionals, prove that—

(1) Their reciprocals are proportionals.

(2) The greatest and least of them together are greater than the other two together.

139. If four quantities are proportionals, and the second of them is a mean proportional between the third and fourth, prove that the third will be a mean proportional between the first and second.

140. If $A \propto B^2$, B^3 as C^4, C^5 as D^6, and D^7 as E^4, show that $\frac{A}{E} \times \frac{B}{E} \times \frac{C}{E} \times \frac{D}{E}$ does not vary at all.

141. Two numbers whose sum is $2a^8$ are in the ratio of $a+b-\frac{ab}{a+b}$ to $a-b+\frac{ab}{a-b}$ to each other, find their values.

142. If $x^2 \propto y^2$ and $x=2$ when $y=3$, find the equation between x and y.

143. If 13 gold coins and 12 silver ones are worth three times as much as 3 gold and 40 silver coins, find the ratio which expresses the value of a gold coin to a silver one.

144. Show that $\frac{ax+by}{ay+bx}$ is greater than the least and less than the greater of the fractions $\frac{x}{y}$ and $\frac{y}{x}$.

145. If $a : b :: (a+c)^2 : (b+c)^2$, prove that c is a mean proportional between a and b.

146. If x is small compared with a, show that the ratio of $a^n : (a+x)^n$ is equal to the ratio of $a : a+nx$ very nearly, whether n be integral or fractional.

147. If $\frac{\sqrt{x}-2\sqrt{y}}{\sqrt{x}+2\sqrt{y}}=\frac{\sqrt[4]{x}-2\sqrt[4]{x-y}}{\sqrt[4]{x}+2\sqrt[4]{x-y}}$, show that

$$\frac{x}{y}=\frac{1}{2}(1 \pm \sqrt{5}).$$

148. If x be small compared with a, show that

$$\frac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}}=\frac{x}{2a}.$$

149. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$ etc., show that $\frac{a}{b}=\sqrt{\frac{a^2+c^2+e^2+\text{etc.}}{b^2+a^2+f^2+\text{etc.}}}$

150. Show that whatever be the value of x and y, $\frac{x}{y}+\frac{y}{x}$ is greater than 2.

PERMUTATIONS AND COMBINATIONS.

151. Find the number of combinations of n things taken r at a time. How many words of 3 letters each can be formed of 20 consonants and 5 vowels, the vowel being always supposed to be the middle letter of the word?

152. How many different arrangements can be made of the letters of the alphabet taken three at a time, two consonants and one vowel being in each arrangement?

153. Show that the number of combinations of n things taken r together is the same as the combinations of n things taken $n-r$ together.

154. Given ten white and ten black balls; in how many different ways can I select from them a set of ten balls of which five shall be white and five black?

155. A committee of 7 members is to be chosen out of a body composed of 20 Protestants and 15 Catholics in such a way that there shall be 3 of one creed and 4 of the other. In how many different ways can such a committee be constituted?

156. How many different arrangements can be made with n letters $a, b, c \ldots$? In how many of these arrangements will a and b be next to each other? In how many will a come before b (not necessarily immediately before b).

157. With 17 consonants and 5 vowels, how many words can be formed, with two different vowels in the middle, and one consonant (repeated or different) at each end?

158. In how many ways can I select two white balls and three red out of an urn containing 7 white balls and 10 red?

159. Twelve balls are to be separated into three heaps of 3, 4, and 5 each; in how many ways can this be effected?

160. With 5 dice, how many throws are possible? How many throws in which no two dice shall show the same number of eyes, and, distinguishing the different dice, in how many ways may one of the latter throws be made?

161. An alphabet being supposed to contain m consonants and n vowels, required the entire number of different words, each containing p of the former and q of the latter, that could be formed out of its letters.

162. Three persons have 4 coats, 5 vests, and 6 hats between them, in how many different ways can they dress themselves with them?

163. The operatives in a factory being supposed to consist of a men, b women, c boys, and d girls, required the entire number of different combinations of p men,

q women, r boys, and s girls that can be told off from among them to any particular work.

164. The number of combinations of n things taken p together + the number of n things taken $(p-1)$ together = number of $(n+1)$ things taken p together.

165. A telegraph has m arms, and each arm is capable of n distinct positions, find the total number of signals that can be made with the telegraph.

166. Find the ratio of the number of combinations of $4n$ things taken $2n$ together to the combinations of $2n$ things taken n together.

167. How many different sums of money can be formed with a farthing, a halfpenny, a penny, a sixpence, a shilling, a crown, a half-sovereign, a sovereign, and a guinea?

168. Into how many different triangles may a decagon be divided by lines from its angular points?

169. If the number of combinations of $(n+1)$ things taken 4 together = 9 times the combinations of n things taken 2 together, find n.

170. Out of 12 workmen, 8 are always required to work together. How many sets of 8 can be arranged without the same set recurring?

ARITHMETICAL PROGRESSION.

171. The sum of 40 terms of an A. P. is a, and the sum of 50 terms is b. Find the common difference.

172. Prove the rule for finding the sum of n terms of an arithmetical progression.

173. In former times troops for the purpose of making a stand on all sides used to be drawn up in the form of a solid triangle, 1 man in the first rank, 3 in the second, 5 in the third, and so on. Prove that a triangular battalion so formed would always admit of being transformed into a solid square.

174. If the squares of three quantities be in A. P., so also will be the reciprocal of their sums taken two and two and two together. Prove this, and give a numerical illustration.

175. Having given the first term, the common difference, and the sum of an A. P., find the number of terms.

176. If the first term is 27, the fourth 18, and the sum 117, find the number of terms and the last term.

177. If $y=ax+b$, show that the values which y assumes, when values in arithmetical progression are substituted for x, are themselves in arithmetical progression.

178. Given that in an arithmetical progression of n terms, commencing with unity, the sum is equal to the number of terms, determine the number of terms.

179. There are m arithmetical progressions, each beginning with unity, and their common differences are 1, 2, 3, etc. . . . m. Show that the sum of their nth terms $=\frac{1}{2}\{(n-1)m^2+(n+1)m\}$.

180. A body of soldiers are drawn up in the form of a hollow equilateral wedge, the ranks of which are three deep, the outer rank containing n persons. Find the number of men.

GEOMETRIC PROGRESSION.

181. Find the sum of n terms of a G. P., having given the first term and common ratio. Find also the sum of the products of every pair of different terms.

182. Find the sum of n terms of a G. P., having given the first and last terms and the number of terms.

183. The first term of a G. P. is 3, and the fourth term is $\frac{1}{\sqrt{3}}$; find the sum to infinity.

184. Sum the series $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+$etc. to n terms, and show that the sum of any odd number of terms of this series is always greater, and the sum of any even number of terms always less than the sum to infinity. What is the least number of terms of the series which will give a sum differing from the sum to infinity by less than $\cdot 0001$?

185. What is the value of the series $\frac{1}{2}-\frac{1}{3}+\frac{2}{9}-\frac{4}{27}$ +etc. *ad infinitum*?

186. The arithmetical mean between two numbers is $1+a^2$, and the geometric mean $1-a^2$; what are the numbers?

187. Three numbers are in G. P., the common ratio is equal to the first, and also equal to nine-tenths of the sum of the second and third. Find the three numbers.

188. Find the sum of $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+$etc. to infinity,

and the sum of the least number of terms of the series differing by less than $\frac{1}{1000}$ from the sum to infinity.

189. Find the sum of

$$\frac{1}{8}+\frac{4}{8^2}+\frac{6}{8^3}+\frac{3}{8^4}+\frac{1}{8^5}+\frac{4}{8^6}+\frac{6}{8^7}+\frac{3}{8^8}+\frac{1}{8^9}+\text{etc.}$$

to infinity.

190. In a G. P. of n terms commencing with unity, the sum is a number consisting of n digits, each equal to unity. Determine the common ratio.

191. Sum to infinity $\frac{\sqrt{2}+1}{\sqrt{2}-1}+\frac{1}{2-\sqrt{2}}+\frac{1}{2}+\text{etc.}$

192. If $\cdot abbb$, etc., be a decimal where a contains m places, and b, n places, show that

$$\cdot abbb, \text{ etc.} = \frac{a.10^n+b-a}{10^m(10^n-1)}.$$

193. Sum the series, $1+2a+2a^2+2a^3+\text{etc.}$, to infinity.

194. If the sum of a series to infinity $=2$, and the sum of the squares of the terms of the same series $=\frac{4}{3}$, find a and r.

195. If $\frac{m+nx}{m-nx}=\frac{n+px}{n-px}=\frac{p+qx}{p-qx}$, etc., show that m, n, p, q, etc., are in G. P.

196. Show that the sums of the two series $1+\frac{3}{2}+\frac{5}{4}+\frac{7}{8}+\text{etc.}$, and $1-\frac{3}{2}+\frac{5}{4}-\frac{7}{8}+\text{etc.}$, to infinity, are as $27:1$.

197. In a G. P. if $r=\frac{1}{2}$, any term is equal to the sum of all the terms that follow it. Prove this.

198. Find the sum of twenty terms of the following series :—

$$(\alpha)\ \frac{x-1}{10}+\frac{x-2}{10}+\frac{x-3}{10}+\text{etc.}$$

$$(\beta)\ \frac{1}{\sqrt{2}}-1+\sqrt{2}-2+2\sqrt{2}-3+, \text{ etc.}$$

199. Find the sum of the series whose nth terms are

(α) 2^n+4n.

(β) $\sqrt{2})^n-n$.

(γ) $3n-1$.

200. Find the sum of the series, 2, 3, 5, 11, 25, to n terms.

EQUATIONS.

201. $\frac{1}{x-1}+\frac{2}{x-2}=\frac{3}{x-3}$.

202. $\frac{9x+20}{36}=\frac{4x+20}{35x+2}+\frac{x}{3}$.

203. $\frac{x-6}{10}+\frac{x+3}{5}=x-\frac{7}{10}$.

204. $\dfrac{x+\frac{1}{x}-1}{x-\frac{1}{x}+1}=1-\left(x-\frac{1}{x}\right)$.

205. $\frac{x^2+x+\frac{1}{2}}{a^2+1}+\frac{x^2+x}{a^2-1}=0$.

206. $\frac{x-1}{x+1}+\frac{x+1}{x-1}=\frac{5x}{x^2-1}$.

207. $\frac{2x^2-3x+1}{x^2-2x+2}=\frac{2x-3}{x-2}$.

208. $2x^2-21x+55=0$.

209. $\sqrt{x^2-8x+31}+(x-4)^2=5$.

210. $\frac{x-1}{x-2}-\frac{x-2}{x-3}=\frac{x-4}{x-5}-\frac{x-5}{x-6}$.

211. $\frac{x+a}{x-a}+\frac{x+b}{x-b}+\frac{x+c}{x-c}=3$.

212. $\frac{7x^n}{x-1}=\frac{6x^{n+1}+x^n}{x+1}-\frac{3x^n+6x^{n+2}}{x^2-1}$.

213. $\frac{3abc}{a+b}+\frac{a^2b^2}{(a+b)^3}+\frac{(2a+b)b^2x}{a(a+b)^2}=3cx+\frac{bx}{a}$.

214. $h\sqrt[3]{ax-b}=k\sqrt[3]{cx+dx-f}$.

215. $\sqrt[3]{a^2+c}=\sqrt[4]{\frac{a^2+c}{d(x+g)}}$.

216. $\dfrac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}}=m$.

217. $\sqrt[n-m]{a+x}=\sqrt[2m]{x^2+5ax+b^2}$.

218. $1+x=\sqrt{1+x\sqrt{16+x^2}}$.

219. $\sqrt{\frac{a}{x}} = \sqrt{\frac{a}{x} + \sqrt{\frac{b^2+x^2}{a^2}}} - \sqrt{\frac{x}{a}}.$

220. $\frac{ax}{b}\sqrt{f^2x^2+d^2} + afx^2 = cx.$

221. $a^2(x^2-1) - 2(x+1)^2 = \frac{1}{a^2}(1-4a^2x-x^2).$

222. $\frac{x\sqrt{64x-528}}{\sqrt{3+x}} = 8(x-6).$

223. $\frac{1}{x+\sqrt{18-x^2}} + \frac{1}{x-\sqrt{18-x^2}} = \frac{x}{16}.$

224. $\frac{1}{x+\sqrt{a-x^2}} + \frac{1}{x-\sqrt{a-x^2}} = \frac{x}{b}.$

225. $a^2+x^2 = b^2+b^2.\frac{(ac-x)^2}{(a+cx)^2}.$

226. $e^x+e^{-x} = 5\cdot2$, where $e = 2\cdot71828$.

227. $\frac{x-\sqrt{x}}{x+\sqrt{x}} = \frac{36}{x^2-x}.$

228. $\sqrt{2x+7} + \sqrt{3x-8} = \sqrt{7x+1}.$

229. $\sqrt{x+\sqrt{x}} + \sqrt{x-\sqrt{x}} = 3\sqrt{\frac{x}{x+\sqrt{x}}}.$

230. $x\sqrt{x^4-1} + \sqrt[4]{x^4-1} = x^3.$

EQUATIONS OF MORE THAN ONE UNKNOWN QUANTITY.

231. $\frac{1}{x}+\frac{1}{y} = \frac{x+y}{12} = \frac{7}{x+y+5}.$

232. $x^2+xy = a.$
$y^2+xy = b.$

233. $\frac{x}{a}-\frac{y}{b} = 1+\frac{a^2}{b^2}.$
$\frac{x}{b}+\frac{y}{a} = 1+\frac{b^2}{a^2}.$

234. $x+y = \frac{1}{x}+\frac{1}{y} = \frac{5}{2}.$

235. $xy = 12. \quad x^5-y^5 = 781.$

236. $\frac{x}{a}+\frac{y}{b} = 1-\frac{x}{c} = 1+\frac{y}{c}.$

237. $\dfrac{x}{a+2}+\dfrac{y}{a}=1.\quad \dfrac{x}{a}+\dfrac{y}{a-1}=1.$

238. $\dfrac{x-y}{xy}=\dfrac{1}{6}.\quad \dfrac{x-z}{xz}=\dfrac{2}{3}.\quad \dfrac{y+z}{yz}=\dfrac{3}{2}.$

239. $2x+y+z=a.$
$x+2y+z=b.$
$x+y+2z=c.$

240. $\dfrac{x^2}{y}-\dfrac{y^2}{x}=28.\quad x-y=8.$

241. $\left.\begin{array}{l}\dfrac{x}{b+c}+\dfrac{y}{c-a}=a+b\\[2mm] \dfrac{y}{c+a}+\dfrac{z}{a-b}=b+c\\[2mm] \dfrac{z}{a+b}+\dfrac{x}{b-c}=c+a\end{array}\right\}.$

242. $\left.\begin{array}{l}\dfrac{1}{x+\dfrac{1}{y-\dfrac{1}{x}}}=\dfrac{1}{x-\dfrac{1}{y-\dfrac{1}{x}}}\\[4mm] \dfrac{1}{y}\left(1-\dfrac{1}{x}\right)=1\end{array}\right\}.$

243. $\left.\begin{array}{r}3x+9y-6z=8\\ 4x-7y+13z=9\\ 12x-3y-5z=11\end{array}\right\}.$

244. $\left.\begin{array}{l}x^2+xy=(a-b)^2\\ y^2+xy=4ab\end{array}\right\}.$

245. $\left.\begin{array}{l}ax+by+cz=p^2\\ fx+gy+hz=q^2\\ x^2+y^2+z^2=r^2\end{array}\right\}.$

246. $x^2-xy=10.\quad x-y=2.$

247. $xy^2+xy=12.\quad x(1+y^3)=18.$

248. $x-y=a.\quad ay+bx+y^2=0.$

249. $xy+y=10.\quad x^2y^2+y^2=68.$

250. $y+x\sqrt{y}=21.\quad x^2y+y^2=225.$

251. $\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{5}{6}.\quad \dfrac{1}{x^2}+\dfrac{1}{y^2}=\dfrac{13}{36}.$

252. $x^{\frac{1}{2}}+y^{\frac{1}{5}}=7.\quad x^2+y^{\frac{4}{5}}=641.$

253. $\dfrac{xyz}{x+y}=4\tfrac{4}{5}.\quad \dfrac{xyz}{y+z}=3\tfrac{3}{7}.\quad \dfrac{xyz}{x+z}=4.$

254. $\sqrt{x}^{\sqrt[4]{x}+\sqrt[4]{y}}=(\sqrt{y})^{\frac{8}{8}}.\quad \sqrt{y}^{\sqrt[4]{x}+\sqrt[4]{y}}=(\sqrt{x})^{\frac{2}{3}}.$

255. $x+y=4.\quad x^6-y^6=728.$

256. $xy=3.\quad x^6-y^6=728.$

257. $x^6-y^6=728.\quad x^2+y^2=10.$

258. $x^4=ax+by.\quad y^4=ay+bx.$

259. $xy=x^2-y^2.\quad x^2+y^2=x^3-y^3.$

260. $x-y=1.\quad x^5-y^5=b\,.\,1.$

261. $\sqrt{x^2-y^2}+2(x-y)=2(x-1).$
$15(x^2+y^2)=34xy.$

262. $x+y=2\sqrt{xy}.\quad x+2y=3\sqrt{\dfrac{x}{y}}.$

263. $\sqrt{x+y}=2-\sqrt{x-y}.$
$\sqrt{x^2+y^2}=5-\sqrt{x^2-y^2}.$

264. $x^2+y^2=8.\quad \dfrac{x^2+y^2}{x^2y^2}=\dfrac{1}{2}.$

265. $x+y+z=12.$
$x+2y+3z=26.$
$x^2+y^2+z^2=50.$

266. $x^4+x^2y^2+y^4=133.$
$x^3+y^3=35.$

267. $x^4+x^2y^2+y^4=133.$
$x^2+xy+y^2=19.$

268. $x^2+xy+y^2=91.$
$x^2-xy+y^2=31.$

269. $x^2+y^2=9-(x+y).$
$xy=1.$

270. $x+y+\sqrt{x+y}=12.$
$xy=18.$

271. $x^{x+y}=y^{a+b}.$
$y^{x+y}=x^{4(a+b)}.$

272. $\dfrac{x^2+y}{x-y}=11.\quad \dfrac{x^2-y}{x+y}=\dfrac{7}{5}.$

273. $x^4+y^4=14x^2y^2.\quad x+y=6.$

274. $x+y+z=6.\quad xy+xz+yz=11.$
$xyz=6.$

275. $x^2+xy+y^2=91.$
$y^2+yz+z^2=127.$
$z^2+xz+x^2=109.$

276. $a^x=b.$

277. $a^{mx}b^{nx}=c.$

278. $a^{mx+f}b^{nx+g}=c^{px+h}d^{qx+k}.$

279. $3^x=177147.$

280. $5^{x+1}+25^x=50.$

281. $(x^2+1)(y^2+1)=10.$
$(x+y)(xy-1)=3.$

282. $(x+y)(x+z)=a^2.$
$(y+z)(y+x)=b^2.$
$(z+x)(z+y)=c^2.$

283. $x^3+y^3+z^3=3xyz.$
$3a-x+z=3b-y+x=3c-x+y.$

284. $x^2+y^2-z^2=(x+y-z)^2+2.$
$x^3+y^3-z^3=(x+y-z)^3+9.$
$x^4+y^4-z^4=(x+y-z)^4+29.$

285. $x^2+y^2+z^2=38.$
$2x+3y+5z=29.$
$15x^2+10y^2+6z^2=12xz+12yz+297.$

286. $\dfrac{x+y}{4}=\dfrac{xy}{x+y}+\dfrac{1}{2}.$
$x^2+y^2=\dfrac{8x^2y^2}{(x+y)^2}+22.$

287. $3x+3y-z=3.$
$x^2+y^2-z^2=\dfrac{14-9z}{2}.$
$x^3+y^3+z^3=3xyz+\dfrac{17z+14}{4}.$

288. $\sqrt[3]{x^2+y^2+c^2}+\sqrt[3]{(x-y+c)^2}=2\sqrt[3]{(4xy)}.$
$\dfrac{1}{y}=\dfrac{1}{x}+\dfrac{1}{c}.$

289. $\sqrt[3]{x+y}+\sqrt[3]{x-y}=\sqrt[3]{a}.$
$\sqrt[3]{x^2+y^2}+\sqrt[3]{x^2-y^2}=\sqrt[3]{a^2}.$

290. $x^3=31x^2-4y^2.$
$y^3=31y^2-4x^2.$

291. $7^{\frac{x}{2}+\frac{y}{3}}=2401.$
$6^{\frac{x}{4}+\frac{y}{2}}=1296.$

292. $x^2-y^2=\dfrac{4ab}{a^2-b^2}.$
$\dfrac{2a}{x^2+y^2}+\dfrac{b(x^2-y^2)}{x^2+y^2}=a.$

293. $x(12-xy)=y(xy-8).$
$xy(2y+3x-xy)=24(x+y-4).$

294. $(x^2+y^2)^2-xy(x^2+y^2)=91.$
$x^4+x^2y^2+y^4=133.$

295. $2xy=4(3x+2y)$.
$3xy=4(2x-3y)$.

296. $x^2-3xy=-y^2$.
$x^5+y^5=2$.

297. $7-10y\sqrt{\frac{x}{y}}=xy-16y$.

$$5+(\sqrt{x}-3\sqrt{y})^2=\frac{9x^2}{64}+2\sqrt{y+2}.$$

298. $3y(x^2+y^2)=26x$.
$2x(x^2-y^2)=15y$.

299. $y(x-2)+x=2y^2+\sqrt{xy}(y^2-1)$.
$xy(xy-18)=4\sqrt{xy}-48$.

PROBLEMS.

300. A horse is sold for £24, and the number expressing the profit per cent. expresses also the cost price of the horse. Find the cost price.

301. Two partners, A and B, trade together, the share of A being twice that of B. Having agreed to dissolve partnership, A takes one-quarter of the stock-in-trade, B takes the remainder and pays A £500. What is the stock-in-trade worth?

302. A rectangular court is 10 yards longer than it is broad, and its area is 1131 square yards. What is its length and breadth?

303. An express train which ought to travel at a uniform speed, after being an hour in motion was delayed half an hour by an accident, after which it proceeded at three-fourths of its original rate of speed, and in consequence arrived at the end of its journey 1 hour 50 minutes late. Had the accident occurred (and the same delay and subsequent retardation taken place) after the train had travelled a distance of 60 miles, it would have been 1 hour 40 minutes late. Find the length of the line.

304. Supposing the above question were varied in the latter part of it by your being informed that 'had the accident occurred when the train had gone half way, it would have arrived 1 hour 20 minutes late,' would that information have been correct, and would it have enabled you to determine the length of the line?

305. A man invests £10,000 in land; he borrows $\frac{5}{7}$ of the value, which he invests as before; he again borrows $\frac{5}{7}$ of the value of this investment, and so on

continually. What would be the aggregate amount borrowed if this process were continued indefinitely?

306. The sum of three numbers in A. P. is 33, and the sum of their squares 435. Find the common difference.

307. Find the condition that a quadratic equation $ax^2+2bx+c=0$ may have equal roots; and this condition being satisfied, find the roots. Write down the quadratic equation which has the roots $\alpha+\beta$ and $\alpha-\beta$.

308. The sum of two numbers is 47·437, and their product 486·7641. Determine them correctly to three places of decimals.

309. Out of a cask containing 360 quarts of pure alcohol a quantity is drawn off and replaced by water. Of the mixture a second quantity, 84 quarts more than the first, is drawn off and replaced by water. The cask now contains as much water as alcohol. Find how many quarts were drawn off the first time. Show that the problem has only one solution.

310. The sum of two numbers is 1878, and their product is 880821. What are the numbers?

311. Given that $a_1x^2+2h_1x+b_1$ and $a_2x^2+2h_2x+b_2$ have a common factor, prove the equation of condition $(a_2b_2+a_2b_1-2h_1h_2)^2=4(h_1{}^2-a_1b_1)(h_2{}^2-a_2b_2)$.

312. The product of two numbers is 180, but if the less be increased by 1, the product is increased by 20. What are the numbers?

313. The cube root of a number is one-fifth of the square root. Find the number.

314. Find two numbers, such that the sum, product, and difference of their squares may be equal.

315. There are three numbers, and the product of the squares of the first and second divided by the third number is 8; the product of the squares of the first and third divided by the second is 64; and the product of the squares of the second and third divided by the first is 512. Find the numbers.

316. The sum of two numbers multiplied by the sum of their cubes $=112$, and the cube of their sum : their difference :: 32 : 1. Find the numbers.

317. There is a number of three digits, of which the sum of the squares $=104$; the square of the middle digit exceeds twice the product of the other two by 4, and if 594 be taken from the number, the digits will be inverted. What is the number?

318. A person bought a number of pigs for £80; if he had bought 4 more for the same money, he would have paid £1 less for each. How many did he buy?

319. Find two numbers whose difference is 3, and the difference of their fourth powers is 2145.

320. The number of cubic feet in a cubical vessel is greater by 4 than the number of square feet in its base. How many cubic feet does it contain?

SECTION III.—GEOMETRY AND TRIGONOMETRY.

321. The sides about the equal angles of triangles which are equiangular to one another are proportionals, and those which are opposite to the equal angles are homologous sides—that is, the antecedents and consequents of ratios.

322. Two circles are given in position and magnitude. If two parallel straight lines be drawn, each touching one of the circles, the straight line which joins the point of contact passes through one of two fixed points.

323. On a given straight line describe a rectilinear figure similar and similarly situated to a given rectilinear figure.

324. Let AB be a given straight line, and O a given point not in the straight line. Join O to any point P in AB, and draw OQ perpendicular to OP, and equal to twice its length. Find the path traced out by Q as P moves along AB.

325. If two straight lines be parallel, and one of them be at right angles to a plane, the other shall also be at right angles to the same plane.

326. If two triangles have one angle of the one equal to one angle of the other, and the sides about the equal angles proportionals, the triangles shall be equiangular to one another, and shall have those angles equal which are opposite to homologous sides.

327. If the perpendicular drawn from the vertex of a triangle to the base be a mean proportional between the segments of the base, show that the triangle is right-angled.

328. Parallelograms about the diameter of any parallelogram are similar to the whole parallelogram and to one another.

329. ABCD is a parallelogram, F is any point in the diagonal AC; through F a straight line is drawn parallel to AB, and meeting AD at G and BC at H, and through F another straight line is drawn parallel to BC, meeting AB at E and CD at K. Show that EH, AC, and GK being produced meet in a point.

330. If two straight lines be cut by parallel planes, they shall be cut in the same ratio.

331. If a straight line be drawn parallel to one of the sides of a triangle, it shall cut the other sides or those sides produced proportionally; and if the sides or the sides produced be cut proportionally, the straight line which joins the points of section shall be parallel to the remaining side of the triangle.

332. Find a point within a triangle, the perpendiculars from which, on the sides, are proportional to those sides.

333. Show how to draw a straight line perpendicular to a plane from a given point without it.

334. Show that all straight lines drawn from an external point to touch a given sphere are equal.

335. If a pyramid be such that it can have a sphere inscribed touching its six edges, prove that the sum of every two opposite edges is the same.

336. Define similar figures, and prove that two rectangles are similar if two adjacent sides of the one are to one another as two adjacent sides of the other.

When are two similar figures said to be similarly situated?

337. Similar triangles are to one another in the duplicate ratio of their homologous sides.

338. Show that similar triangles are to one another as the squares of the radii of the circles inscribed in them.

339. If a straight line is at right angles to each of two straight lines at their point of intersection, it shall also be at right angles to the plane in which the two straight lines lie.

340. Prove the following properties of the sphere:—(1) Every plane section of a sphere is a circle. (2) If from any point outside a sphere straight lines be drawn touching the sphere, their points of contact with the sphere shall lie on a small circle of the sphere.

341. Prove that if any angle of a triangle be bisected by a right line produced to cut the opposite side, the

segments into which that side is divided will have the same ratio to one another as the sides containing the angle. What relation connects the three pairs of segments into which the sides of a triangle are divided by lines bisecting the opposite angles?

342. Prove that if the sides about each of the angles of two triangles are proportional, the triangles will be similar.

343. Give an easy construction for determining any number of polygons similar and similarly situated to a given polygon.

344. When is a right line said to be perpendicular to a plane? What is the measure of the inclination of a right line to a plane? of two planes to each other? and of two right lines, which do not meet, to each other?

345. If a solid angle is contained by three plane angles, prove that the sum of any two is greater than the third.

346. Define a sphere, a cone, a cylinder, a right cone, and a right cylinder. Prove that a hollow cone into which a sphere fits is a right cone.

347. If two circles touch externally, and a common tangent be drawn at the point of contact, this tangent shall bisect any other tangent which is common to both circles.

348. Prove that if the four sides of any quadrilateral figure are bisected, the four points of bisection are the four vertices of a parallelogram of which the area is one-half of the area of the quadrilateral figure.

349. In a right-angled triangle the rectilinear figure described on the side opposite to the right angle is equal to the similar and similarly situated figures on the sides containing the right angle.

350. If two planes which cut one another be each of them perpendicular to a third plane, their common section shall be perpendicular to the same plane.

351. If two spheres intersect, the curve of intersection is a circle of which the plane is at right angles to the line joining the centres of the two spheres.

352. Given a circle, and a point not in the plane of the circle; find the centre of the sphere which passes through the given point and through the circumference of the given circle.

353. OA, OB, OC are three adjacent edges of a cube.

Given $OA = OB = OC = a$. Find the solid contents of the pyramid OABC, and the area of the triangle ABC.

354. Prove that equiangular parallelograms bear to one another the ratio which is compounded of the ratios of their sides.

355. If ABC is a triangle, and a line is drawn bisecting, either internally or externally, the angle at C, and meeting AB or AB produced in D, state what relation exists between the four distances AC, BC, AD, BD, and the angle which the two bisectors include between them.

356. Prove that if X is any point whose distances from two fixed points A, B are in a given ratio, X will always be found in a circle whose centre is in the line AB produced, and find its position therein.

357. Draw a straight line perpendicular to a plane from a given point without it.

358. If a plane is at right angles to a straight line, prove that it is at right angles to every plane passing through the line.

359. Starting from the geometrical rule for finding the area of a triangle, obtain an expression for the surface of a cone in terms of its height and the radius of its base.

360. State the relation between the respective volumes of a prism and pyramid having the same base and altitude. Show that any triangular prism may be divided into three equal pyramids, having for a common edge any one of the six diagonals lying on the three rectangular faces of the prism.

361. Similar polygons may be divided into the same number of similar triangles, having the same ratio to one another that the polygons have.

362. From a fixed point a series of straight lines are drawn to cut two parallel straight lines. Prove that the segments into which the one is thus divided are proportional to the corresponding ones of the other; and conversely, that straight lines which cut two parallel straight lines proportionally, meet in a point.

363. Prove that the straight lines which join the vertices of a triangle to the middle points of the opposite sides meet in a point. Prove, also, that this point divides each of the lines into segments which bear to each other the ratio of 1 : 2.

364. Two planes are parallel if two straight lines

which meet in the one are parallel respectively to two straight lines in the other.

365. Given the radius of a cylinder, find the altitude such that the area of the whole surface may be four times that of the base.

366. Find also the altitude of a cone having the same base as the cylinder, such that the area of its curved surface may be equal to that of the curved surface of the cylinder; and determine the ratio of the volume of the cylinder to that of the cone.

367. The angles or sides of two rectilinear triangles being supposed to correspond in pairs, show that when the three pairs of corresponding angles are equal, the three pairs of opposite sides are proportional; and conversely, that when the three pairs of corresponding sides are proportional, the three pairs of opposite angles are equal.

368. When two rectilinear triangles are similar as solid figures, show that their volumes are to each other in the duplicate ratios of the lengths of their homologous sides.

369. When two triangular pyramids are similar as solid figures, show that their volumes are to each other in the triplicate ratio of the lengths of their homologous sides.

370. When three right lines start from the same point, but do not lie in the same plane, show that the sum of the angles made by any two of them with the third is greater than the angle they make with each other.

371. When, of two great circles of a sphere, either passes through the two poles of the other, show that reciprocally the latter passes also through the two poles of the former.

372. Prove that two isosceles triangles, with their vertical angles equal, have their altitudes proportional to their bases.

373. Construct a triangle similar to a given triangle, and having a given perimeter.

374. If two straight lines which meet are divided proportionally by a series of secants, prove that the latter must be parallel.

375. Find the locus of points in a plane equidistant from a given point without the plane.

376. Prove that two triangular pyramids with equal bases and equal altitudes are equal in volume.

377. A right cone has the same superficial area as the hemisphere on the same base. Show that every plane passing through its axis intersects it in an equilateral triangle.

378. Given two finite right lines, one divided into a number of segments, and the other undivided, show how to divide the latter similarly to the former.

379. Given two rectilinear figures, show how to construct a third which shall be similar to one of them and equal to the other.

380. Given a plane and a point without it, show how to draw through the point a straight line which shall be perpendicular to the plane.

381. Two perpendiculars are drawn from a point to two intersecting planes, show that whatever be the position of the point, the plane of connection of the perpendiculars is perpendicular to the line of intersection of the planes.

382. An equilateral triangle being supposed to revolve on its altitude as an axis, show that the area of the cone generated by either of its sides is double that of the circle generated by either half of the base.

383. Three cylinders of equal altitude have for their bases the three circles described on the three sides of a right-angled triangle as diameters. Show that the volume of the greatest is equal to the volumes of the other two.

384. A cone and a hemisphere have a common base, and are on opposite sides of it. What is the ratio of the altitude of the cone to the radius of the hemisphere, if the volumes of both solids are equal?

385. Prove that two triangles are similar, if the sides of the one are respectively parallel to those of the other.

386. If two similar rectilinear figures are situated in such a manner that two sides of the one are respectively parallel to the two homologous sides of the other, prove that every side of the one is parallel to the homologous side of the other, and that the lines joining homologous vertices all pass through a common point.

387. On a straight line three points A, B, C are given, B lying between A and C. Find a point D which divides the line AC externally in the same ratio as B does internally.

388. Two angles in space have the limits of the one parallel respectively to those of the other. Prove that the

angles are equal or supplemental, and that the planes are parallel.

389. Determine the radius of the circle in which a plane cuts a sphere whose radius is 13 inches, if the distance of the plane from the centre of the sphere is 12 inches.

390. The altitude of a right cone equals the circumference of its base. Calculate the volume and the total of the cone, the radius of the base being given.

391. If a cubic foot of ivory weigh 1820 ounces avoirdupois, what will be the loss of weight in turning an ivory ball out of a cube whose edge is 1·25 inches?

392. Find the perpendicular height of a triangular pyramid whose solid content is 500 cubic feet and each side of the base 10 feet.

393. A cubic foot of brass is to be drawn out into a wire one-fortieth of an inch in diameter; what will be the length of the wire, allowing no loss by waste or compression?

394. Find the solidity of a hexagonal pyramid, each side of the base being 10 inches, and each of the slant edges from the vertex to the corners of the base is $12\frac{1}{2}$ inches.

395. How many gallons of water will be required to fill a cubical cistern measuring 3 ft. $2\frac{1}{2}$ in. every way, reckoning $277\frac{1}{4}$ cubic inches to a gallon?

396. From a glass cylinder 2·4 inches in height and 1·3 in. in diameter, I want to make a triangular prism 3·6 in. in length, and having its base equilateral. If I use all the glass, what will be the side of the equilateral triangle which forms its base?

397. If the length of the side of a cube be 3 inches, what will be the side of a cube of double the solidity?

398. A cylindrical tin vessel is to be made to contain 3 cubic feet of water, and its depth must not exceed 18 inches; what must be the diameter of its base?

399. If a heavy sphere whose diameter is 4 inches be put into a conical glass full of water whose diameter is 5 inches at the brim and depth 6 inches, how much water will run over?

400. What should have been the diameter of the ball in the above example, to permit it to sink just so far into the glass as to be covered by the water and no more?

TRIGONOMETRY.

401. Define the sine and cosine of an angle. Show that, whatever be the magnitude of an angle,

$$\sin A = \cos (90 - A).$$

402. Solve the equations—

(1) $\sin^2 \theta + \cos^2 (90 -) = 1.$
(2) $\tan \theta = 2 \sin \theta.$

403. Show that in any triangle $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$.

404. If the sides of a triangle were given, would this be a convenient formula to find the angle A? and if in any case it is not, what formula should be used?

405. The sides of a triangle are 6, 8, and 10, find the greatest angle.

406. Show how to solve a triangle when two sides and an angle opposite to one of them are given.

407. Explain how you would find the distance between two inaccessible objects on a level plain.

408. Prove that $\tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \cdot \tan B}$.

409. If $A + B + C = 180°$, prove that

$$\sin^2 A = \sin^2 B + \sin^2 C - 2 \sin B \cdot \sin C \cdot \cos A.$$

410. Determine the values of the trigonometrical ratios for an angle of 60°.

411. Find A, B, C from the equations—

$$\cos (A + B - C) = \frac{1}{2},$$

$$\cos (A - B + C) = \frac{\sqrt{3}}{2},$$

$$\cos (A + B) = \sin C.$$

412. Show how to find the height and the distance of an inaccessible object on a horizontal plain.

413. A person standing on the bank of a river observes the angular elevation of the top of a tree on the opposite bank to be 60°; and when he retires 100 feet from the edge of the river, he observes the angle to be 30°. Find the height of the tree and the breadth of the river.

414. In any triangle ABC, the tangent of half the difference of the angles B and C is to the tangent of half their sum as the difference of the two sides AB and AC is to their sum.

If $b=17$, $c=7$, $A=60°$, find B and C, having given

$$\log 2 = 3010300,$$
$$\log 3 = 4771213,$$
$$\log \tan 35° \, 49' = 9{\cdot}8583357,$$
$$\log \tan 35° \, 49' \, 10'' = 9{\cdot}8583800.$$

415. Find an expression for the area of a triangle in terms of its sides.

The sides of a triangle are in arithmetical progression, and its area is four-fifths of that of an equilateral triangle of the same perimeter; show that the sides of the triangle are as the numbers 7, 10, 13.

416. Solve the equations—

$$(1) \ \sin x + \cos x = 1,$$
$$(2) \ \cos 2x = \cos^2 x.$$

417. Show that

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2},$$

and investigate a corresponding expression for $\cos A - \cos B$.

418. Show that—

$$(1) \ \sin (A+B) \sin (A-B) = \sin^2 A - \sin^2 B;$$
$$(2) \ \sqrt{1 - \sin 2A} = \cos A - \sin A.$$

419. Three inaccessible objects, A, B, C, are on a level plain, and their distances are known by means of a map. The angles AOB, BOC, being observed at some place O, show how to find the distances AO, BO, CO by formulæ adapted to logarithmic calculation.

420. Given in a triangle two sides and the included angle, investigate a formula to find the difference of the other two angles of a triangle.

421. In any triangle, show that

$$\tfrac{1}{2}(a^2+b^2+c^2) = bc \cos A + ca \cos B + ab \cos C.$$

422. Define the cosine of an angle; and trace the variations in magnitude and sign of the cosine as the angle increases from 0° to 180°.

423. Show that in any triangle, of which the sides are a, b, c, and the angles opposite to them A, B, C—

$$(\alpha) \ \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c};$$

$$(\beta) \ \tan \left(\frac{A-B}{2}\right) = \frac{a-b}{a+b} \cot \frac{C}{2}.$$

424. A tower 100 feet high is observed at a station A, which is on a level with the base of the tower, and the angle of elevation of the top of the tower is 45°. The

observer then proceeds from A to B in a direction at right angles to the line joining A to the base of the tower; and he finds that at the station B (which is on the same level with A) the angle of elevation of the top of the tower is 30°. What is approximately the distance between the stations A and B?

425. Explain how in analysis an angle is regarded as susceptible of assuming all degrees of magnitude between positive and negative infinity. What are the limits of the magnitude of an angle in Euclidean geometry?

426. From the general expression for the sine of the sum of two angles, deduce the formulæ for the cosine and tangent of their difference.

427. What is meant by the ambiguous case in the solution of triangles? What parts are given when the case arises? These parts being given, is the solution necessarily ambiguous? If not, determine the conditions of the ambiguous case arising.

428. Two cliffs stand facing each other on opposite sides of a river. From the top of one of them, known to be 200 feet high, the angles of depression of the summit and foot of the other are observed to be 30° and 45° respectively. Find in feet and inches the height of the latter and the breadth of the river.

429. Prove the formula, $\sin(A+B)=\sin A \cos B + \cos A \sin B$, drawing the figure for the case in which A and B are each less than 90°, but A + B greater than 90°.

430. Find the sine and cosine of 75°.

431. Given in a plane triangle, a, B, and A, solve the triangle.

432. If the sides of a triangle are 9 feet, 7 feet, and 4 feet, what are the sines of the angles of the triangle?

433. Prove the formula, $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$, and apply it to find the values of sin 18°, cos 36°.

434. The three sides of a triangle being given, obtain formulæ for the sines, cosines, and tangents of the semi-angles. Which of these formulæ is to be preferred in computing the angles by means of logarithms, and why?

435. Given the side of the base of a regular square pyramid and the length of the edge, find its height, and the inclination of any one of its faces to the square base.

436. If the edge and side of the base (in the last example) are equal, prove geometrically that the inclination to each other of any two triangular faces will be measured

by the angle between the lines joining the middle point of any edge with the extremities of the adjacent sides of the base, and calculate its cosine.

437. Prove geometrically that

$$\cos(A-B)=\cos A.\cos B+\sin A.\sin B.$$

438. Given the area of a triangle and two of its sides, show how to find the third side and the angles.

439. Assuming the value of π to five places of decimals, calculate to the nearest integer the number of seconds in the angle subtended at the centre of a circle by an arc of length equal to that of the radius of the circle.

440. Assuming the fundamental formulæ for $\sin(A+B)$ and for $\cos(A+B)$, deduce from them those for $\sin 3A$ and $\cos 3A$ in terms of $\sin A$ and $\cos A$ respectively.

441. Given in a plane triangle the base c, and the two angles A and B, express in terms of them the two sides a and b, the altitude h, and the area.

442. Prove that

$$\cos A+\cos B=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}.$$

443. If A, B, C are the angles of a triangle, show that

$$\sin A.\sin B.\sin C=\sin A.\cos B.\cos C$$
$$+\sin B.\cos C.\cos A+\sin C\cos A\cos B.$$

444. By aid of trigonometrical formulæ obtain an expression for the area of a triangle in terms of its sides.

445. Assuming the formulæ for $\sin(A+B)$ and $\cos(A+B)$, deduce from them the corresponding formulæ for $\tan(A+B+C)$ and $\cot(A+B+C)$.

446. Given of a plane triangle any two sides, a and b, and the angle C, express in terms of them the remaining side c, the two angles A and B, and the area.

447. The three sides a, b, c of a plane triangle are 13, 14, and 15 feet respectively; calculate in square feet its area, and in ordinary fractions the sines of the three angles A, B, C.

448. Determine the sine and cosine of an angle whose tangent $=-2$, and whose sine is positive.

449. A, B, C being the angles of a triangle, show that $\tan A.\tan B.\tan C=\tan A+\tan B+\tan C$.

450. A vertical pole (more than 100 feet high) consists of two parts, the lower being $\frac{1}{3}$ of the whole. From a point in a horizontal plane through the foot of the pole,

and 40 feet from it, the upper part subtends an angle whose tangent is $\frac{1}{5}$. Determine the height of the pole.

451. Show that the area of a triangle

$$=\frac{1}{2}a^2\,\frac{\sin B\,.\sin C}{\sin A}.$$

452. Show that the area of a triangle

$$=\frac{1}{2}a^2\,\frac{\sin B\sin C}{\sin (B+C)}.$$

453. In a right-angled triangle, show that

$$\cos (2A-B)=\frac{a}{c^3}(3c^2-4a^2).$$

454. Prove that the area of a right-angled triangle

$$=S\,(S-c)\text{ where }S=\frac{a+b+c}{2}.$$

455. In an isosceles triangle, A being the vertical angle, show that $\cos B=\dfrac{\sin A}{2\sin C}$.

456. Walking along a straight road, an observer notices that the greatest angle which two objects subtend is α; he then walks on a distance c, and the objects are then in a straight line with him, and their directions make an angle β with the road. Find the distance between the objects.

457. Prove that the area of a triangle

$$=\frac{2abc}{a+b+c}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

458. The sides a, b, c of a triangle are in arithmetical progression; show that its area : that of an equilateral triangle of the same perimeter :: $\frac{1}{b}\sqrt{4ab-4a^2-3b^2}$: 1.

459. Show that $\dfrac{\cot A}{\cot A-\cot 3A}+\dfrac{\tan A}{\tan A-\tan 3A}=1$.

460. Find all the values of θ which satisfy the equation $\cos 2\theta+\sin\theta=1$.

461. If $\tan^2 x=\tan (a-x)\tan (a+x)$, show that $\sin 2x=\sqrt{2}.\sin a$.

462. Find $\tan (A+B)$ when

$$\sin A=\frac{2n}{n^2+1},\ \sin B=\frac{2p}{p^2+1}.$$

463. Show that $\sin 7\frac{1}{2}=\frac{1}{4}(1+\sqrt{2}-\sqrt{3})\,(\sqrt{2-\sqrt{2}})$.

464. If A, B, C be the angles of a triangle, show that $\cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A . \cos B . \cos C = 1$.

465. Prove that $\tan A + \dfrac{1}{\tan A} = \dfrac{2}{\sin 2A}$.

466. Solve the equation—

$$(1 - \tan \theta)(1 + \sin 2\theta) = 1 + \tan \theta.$$

467. Given $\cos 36 = \dfrac{\sqrt{5}+1}{4}$ and $\cos 60 = \frac{1}{2}$, find $\cos 240°$ to three places of decimals.

468. Show that $\dfrac{\sec A + \sec 3A}{\sec A \sec 3A} = \dfrac{2 \sin A}{\tan 2A - \tan A}$.

469. Find the value of θ when $\cos \theta - \sin \theta = \sqrt{2}$.

470. If $\sin A (\sin A + 1) = 1$, show that

$$\cos^2 A (\cos^2 A + 1) = 1.$$

471. Show that

$$(\sin A + \cos A)(\tan A + \cot A) = \sec A + \operatorname{cosec} A.$$

472. The circular measure of the difference of the two acute angles of a right-angled triangle is $\dfrac{\pi}{18}$; express the two angles in degrees.

473. Show that $\tan \dfrac{A}{2} = \dfrac{-1 \pm \sqrt{1 + \tan^2 A}}{\tan A}$.

474. If in a triangle

$$(a^2 + b^2) \sin (A - B) = (a^2 - b^2) \sin (A + B),$$

show that the triangle is isosceles or right-angled.

475. A balloon moving in a horizontal direction has a southern elevation A, and a few minutes afterwards it is observed to be south-west at the same elevation. In what direction is it moving?

476. Two sides of a triangle are 3 and 4 feet respectively. If the angle opposite the latter is $78° 14'$, what is the third side? (Sine $39° 7' = \cdot 630902$.)

477. Find the value of $\tan 4A$ in terms of $\tan A$.

478. Two sides of a triangle are $8 \cdot 18$ and $7 \cdot 62$, and their included angle $58° 12'$. What is the third side? (Sin $29° 6' = \cdot 486335$.)

479. Find the values of $\sec 22\frac{1}{2}°$ and $\operatorname{cosec} 22\frac{1}{2}°$.

480. Show that in any triangle $\sin \dfrac{A}{2} \sin \dfrac{B}{2} \sin \dfrac{C}{2}$ is less than 1, except when A, B, and C are equal.

481. Solve the equation, $3 \sec^4 A + 8 = 10 \sec^2 A$.

482. Show that

$$\cos^6 A - \cos 2A = \sin^6 A - \frac{\cos 2A \sin^2 2A}{4}.$$

483. If, in a triangle, C is an obtuse angle, show that $\tan A$ is less than $\frac{1}{\tan B}$.

484. Show how to solve a triangle, having given the three perpendiculars from the angles on the opposite sides.

485. Show that the area of a triangle

$$= \frac{a^2 - b^2}{2} \cdot \frac{\sin A . \sin B}{\sin (A - B)}.$$

486. Show that the lines which bisect the angles of a triangle meet in a point.

487. Find the radius of a circle inscribed in a triangle.

488. Find the radius of a circle circumscribed about a given triangle.

489. A square pyramid has each side of its base 200 feet long, and each edge 150 feet; find the slope of each face, having given

$$\log 2 = \cdot 30103, \quad \log \tan 26^\circ\ 33' = 9 \cdot 69868,$$
$$\log \tan 26^\circ\ 34' = 9 \cdot 69900.$$

490. Given the base and the difference of the angles at the base, to solve the triangle.

491. Prove that $\cot \frac{1}{2} A = \sqrt{\frac{1 + \cos A}{1 - \cos A}}$.

492. If $2 \cos A = x + \frac{1}{x}$, find $2 \cos 3A$.

493. If $\tan \frac{A}{2} = \frac{\tan A + c - 1}{\tan B + c + 1}$, find $\tan \frac{A}{2}$.

494. Show that

$$\sin A + \sin C - \sin B = 4 \sin \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}.$$

495. Prove that $\frac{\sin (A + B + C)}{\cos A \cos B \cos C}$

$$= 1 - \tan B \tan C - \tan A \tan C - \tan A \tan B.$$

496. Divide a given angle ABC into two angles whose sines shall have a given ratio.

497. Find A when $3 \tan \left(\frac{\pi}{4} - A\right) = \tan \left(A + \frac{\pi}{4}\right)$.

498. Trace the secant in magnitude and sign from 0° to 360°.

499. Trace the changes of sign of $\cos A - \sin A$ from 0 to 180.

500. A person on the slope of a hill observes the angles of elevation, α and β, of two objects on the hill, and also the angle γ which they subtend at his position; if θ be the inclination of the hill to the horizon, show that

$$\sin^2\theta \sin^2\gamma = \sin^2\alpha + \sin^2\beta - 2 \sin\alpha \sin\beta \sin\gamma.$$

ANSWERS.

1. 340 centuries.
2. Factors of $11310 = 2.3.5.13.29$,
   „ „ $86478 = 2.3.29.7.71$,
   „ „ $448630 = 2.5.7.29.211$;
   $\therefore$ 2.3.29 or 174 is the G. C. M. of the first two;
   $\therefore$ 2.7.29 or 406 „ „ of the second two.
   and 2.29 or 58 „ „ of all three.
3. See page 5.
5. See page 7.
6. £24.
7. 72. Let the factors of $N = m.n.p.q$.
   „ „ $M = m.n.p.r$.
   Then G. C. D. = prod. of $m.n.p$.

But $m.n.p$ are the only other divisors, and each of them is a factor of the G. C. D.; $\therefore$ every divisor is contained in G. C. D.

8. 3 seconds.
9. ·304.
10. $\frac{157}{225}$.
11. $220\sqrt{10}$ yards.
12. $33\sqrt{10}$ feet.
13. £12, 1s. 6d. £8, 1s. 0d. £6, 0s. 9d.
14. See page 2.
16. 8 and 6400.
17. $$\frac{\cdot 011 \times 133 \cdot 1 - \cdot 723 \times \cdot 00723}{1 \cdot 1377} = \frac{11 \times 1331 - 7 \cdot 23 \times 7 \cdot 23}{11377}$$
$$= \frac{11^4 - 7 \cdot 23^2}{11377}.$$

The numerator is the difference of two squares, *i.e.* the product of the sum and difference of two quantities

$$= \frac{(11^2 + 7 \cdot 23)(11^2 - 7 \cdot 23)}{11377} = \frac{128 \cdot 27 \times 113 \cdot 77}{11377}$$
$$= \frac{128 \cdot 27}{100} = 1 \cdot 2827. \quad \tfrac{83}{100} \text{ of 9 seconds.}$$

18. $\frac{5295}{74074}$. $\cdot\dot{9}9900\dot{0}$.

19. See page 9.
20. $1\frac{421}{909}$,
21. $\cdot\dot{1}$.
22. $\cdot 196\dot{4}2857\dot{1}$.
23. $\dfrac{\cdot 05^3 + 1}{\cdot 05 + 1} = \cdot 05^2 - \cdot 05 + 1 = \cdot 9525$. $28\frac{4}{7}$ seconds.
24. £250, 1s. 0d.
25. 38,896,200.
26. $\frac{19}{54}$.
27. $\sqrt{10} = 3\cdot 1622$,
$\sqrt{\cdot 004} = \sqrt{\frac{1}{250}} = \frac{1}{5}\sqrt{\frac{1}{10}}$
$= \frac{1}{50}\sqrt{10} = \cdot 063244$.
28. 701 and $\cdot 339\dot{2}8571\dot{4}$.
29. $\cdot\dot{3}$.
30. £150, 0s. 10d.
31. £1561, 9s. $11\frac{11}{16}$d.
32. £100.
33. £450. £900. £1350.
34. £1022.
35. 3 hours $13\frac{7}{11}$ minutes, and 3 hours $19\frac{1}{11}$ minutes.
36. 3·1415.
37. $\frac{2}{5}$.
38. $10\frac{1}{2}$ days.
39. £24, 16s. $2\frac{3}{4}$d. + $\frac{73}{133}$ qrs. $\cdot\dot{0}827067669172932\dot{3}$.
40. 300 of each.
41. 2·71828182.
42. £25·54 and £11·34.
43. 14,390 nearly.
44. £225·7.
45. £18,630.
46. $\cdot 0026684\dot{9}52380\dot{8}$.
47. 230·46. 23·046. 2·3046. ·23046.
48. $\frac{23}{49}$. ·845. 15·4919.
49. $\frac{5}{12}$.
50. 1 hour $5\frac{5}{11}$ minutes, and 1 hour $38\frac{2}{11}$ minutes.
51. See page 26.
52. Number of years $= \dfrac{\log 2}{\log 13 + 3\log 2 - 2} = 17\cdot 7$ years.
53. See page 25.
54. $\bar{3}\cdot 4313638$, and $3\cdot 5475286$.
55. See page 26.
56. See page 21.
57. $\bar{1}\cdot 939794$. $\bar{5}\cdot 30103$. $1\cdot 79588$. $\cdot 650515$.

58. £$6428\frac{4}{7}$.
59. See page 27.
60. ·9754625.
61. See page 27.

63. The number is $10^{\frac{1}{2}} = \sqrt{10} = 3{\cdot}16 +$ etc.
64. £1801, 15s. 0d. nearly.
65. 9. 2·385605.
66. See page 19.
67. Let x = required annuity. Then by question its value is

$$\frac{x}{R-1}\left\{\frac{1}{R^n}-\frac{1}{R^{2n}}\right\}.$$

$$\text{Value of annuity } A=\frac{R^n-1}{R^n(R-1)}A;$$

$$\therefore \frac{x}{R-1}\left\{\frac{1}{R^n}-\frac{1}{R^{2n}}\right\}=\frac{R^n-1}{R^n(R-1)}A,$$

from which $x = AR^n$.

68. See page 20.

69. Present value of £1 due a year hence $=£\frac{1}{1{\cdot}04}$;

$\therefore$ value of annuity of £1 *ad infinitum* $=\frac{1}{1{\cdot}04\times{\cdot}04}$.

Present value of £1 due two years hence $=\frac{1}{(1{\cdot}04)^2}$;

$\therefore$ value of annuity of £1 commencing two years hence

$$=\frac{1}{1{\cdot}04^2\times{\cdot}04}.$$

Similarly, value of annuity of £1 beginning three years hence $=\frac{1}{1{\cdot}04^3\times{\cdot}04}$;

$\therefore$ total value of annuity

$$=\frac{1}{1{\cdot}04\times{\cdot}04}+\frac{1}{1{\cdot}04^2\times{\cdot}04}+\frac{1}{1{\cdot}04^3\times{\cdot}04}\ \textit{ad infinitum}$$

$$=\frac{1}{{\cdot}04}\left\{\frac{1}{1{\cdot}04}\div\left(1-\frac{1}{1{\cdot}04}\right)\right\}$$

$$=\frac{1}{{\cdot}04}\left\{\frac{1}{1{\cdot}04}\div\frac{{\cdot}04}{1{\cdot}04}\right\}=£\frac{1}{{\cdot}04}\times\frac{1}{{\cdot}04}=£625.$$

70. $n=\dfrac{\log 3-\log 2}{2\log 11-\log 10-2\log 2-\log 3}=48{\cdot}8$ years.

71. £$\dfrac{20(1{\cdot}0325^5-1)}{{\cdot}0325\times 1{\cdot}0325^{20}}$. £$\dfrac{10(1{\cdot}01625^{10}-1)}{{\cdot}01625\times 1{\cdot}01625^{50}}$.

72. Find the log of b in the system of which the base is a. Apply formula $\log_a b \times \log_b a = 1$.

73. ·09071416.

74. See page 27.

75. $\text{Log}\ (x \pm \delta) - \log x = \log\left(\frac{x \pm \delta}{x}\right) = \log\left(1 \pm \frac{\delta}{x}\right)$

$$= \pm\mu\left(\frac{\delta}{x} - \frac{\delta^2}{2x^2} + \frac{\delta^3}{3x^3} - \text{etc.}\right).$$

If δ is small compared with x, the right-hand side $= \pm\mu . \frac{\delta}{x}$;

$$\therefore \log\ (x \pm \delta) - \log x = \mu\frac{\delta}{x};$$

$$\therefore \log\ (x \pm \delta) = \log x \pm \mu\frac{\delta}{x}.$$

76. 1·930785.

78. ·6532125135. ·8293037724. 1·0053950313.

79. £71·06.

80. £79·8.

81. £163,800,000 nearly.

82. 47·193 years.

83. £2731, 2s. $5\frac{1}{4}$d.

84. $6·4 = \frac{64}{10} = \frac{2^6}{2 \times 5} = 5 \log 2 - 1 = ·80614$

$$\left(1\tfrac{1}{2}\right)^{20} = \left(\frac{3}{2}\right)^{20} = 20(\log 3 - \log 2) = 3·5218.$$

$$1·25 = \frac{125}{100} = \frac{5^3}{10^2} = 3 \log 5 - 2 \log 10$$
$$= 3(\log 10 - \log 2) - 2 \log 10$$
$$= \log 10 - 3 \log 2 = 1 - 3 \log 2 = ·09691.$$

86. £724, 5s. 0d.

87. $\text{Log}\ (x \pm \delta) = \log x \pm \mu\frac{\delta}{x}$.

Now $x = 1000$, $\delta = 1$;

$$\therefore \log\ (1000 + 1) = \log 1000 + ·43429 \times \frac{1}{1000};$$

$$\therefore \log 1001 = 3 + ·00043429$$
$$= 3·00043429,$$

$$\log\ (1000 - 1) = \log 1000 - ·43429 \times \frac{1}{1000}$$
$$= 3 - ·00043429$$
$$= 2·99956571.$$

88. ·30103, ·477121, ·845098, and 1·623249. See page 19, Ex. 1.

89. Log $2^{64} = 64 \times \cdot 30103 = 19\cdot 26592$. There would be 20 digits.

90. 2·2988531, and 2·303196. See Ex. 87.

91. See Todhunter's *Plane Trigonometry*, Art. 139.

92. £518, 6s. $0\frac{1}{2}$d.

93. £105, 1s. 3d.

94. See page 128.

95. 24·0824.

96. ·846341.

97. ·301030, ·698970, ·397940, 2·096910.

98. £155·82.

99. ·2309306.

100. $\bar{1}\cdot 982093$.

101. $2x^2 - 3xy + 4y^2$.

102. $\dfrac{x^8 - x^4 + 1}{x^8 + x^4 + 1}$.

103. $\dfrac{(x-1)(x-2)}{x+1}$ and $(x+y-a)(x^2+y^2+a^2+ay+ax-xy)$.

104. 0.

105. 1 and 2.

106. $x^2 + 2ax - 3b^2 = x(x-a) + 3a(x-a) + 3a^2 - 3b^2$;

$\therefore x^2 + 2ax - 3b^2$ is divisible by $x - a$,

when $3a^2 - 3b^2 = 0$;

that is, when $a^2 - b^2 = 0$,

or when $a^2 = b^2$,

or $a = \pm b$.

107. (α) $\dfrac{8abc}{(a-b)(b-c)(c-a)}$. ($\beta$) 4.

108. $abxy$ and $1 + 2x + x^2 + 6x^3 + 9x^4$.

109. $3\dfrac{9}{11}$.

110. $2\sqrt{2} + \sqrt{10} + 2 + \sqrt{5}$.

111. 1 and $\dfrac{ax(a-x)^2}{(x^2+ax+a^2)^2}$.

112. $x^2 + \dfrac{x}{2} + \dfrac{1}{2} + \dfrac{1}{x}$.

113. (α) 0. (β) 1. (γ) 0.

114. $\dfrac{2\sqrt{a^2 - b^2}}{b}$.

115. $a^2-b^2+c^2-2ac$.

116. $\dfrac{4ab+2\sqrt{a^4+6a^2b^2+b^4}}{(a+b)^2}$.

117. 6.

118. Add numerators for a new numerator, and denominators for a new denominator.

$$\text{Then } \frac{(a+b+c+d)\ (x+y+z)}{2(a+b+c+d)}.$$

$$\text{If } a+b+c+d \text{ is not} = 0, \text{ this becomes } \frac{x+y+z}{2}.$$

$$\text{If } a+b+c+d=0,\ b+c+d=-a.$$

Let k be the value of each fraction.

$$\begin{aligned}\text{Then } bx+cy+dz&=-2ak,\\ cx+dy+az&=-2bk,\\ dx+ay+bz&=-2ck.\end{aligned}$$

These are clearly satisfied by $x=y=z=2k$, because $b+c+d=-a$.

121. $1+x+x^2+\text{etc.}=\dfrac{1}{1-x}$. The series should be summed first. See page 53.

122. $k^2=x^2+y^2+2x^2y^2+2xy\sqrt{(1+x^2)\ (1+y^2)}$;

$$\begin{aligned}\therefore\ 1+k^2&=1+x^2+y^2+2x^2y^2+2xy\sqrt{(1+x^2)\ (1+y^2)}\\ &=x^2y^2+(1+x^2+y^2+x^2y^2)\\ &\quad+2xy\sqrt{(1+x^2)\ (1+y^2)}\\ &=\{xy+\sqrt{1+x^2+y^2+x^2y^2}\}^2;\end{aligned}$$

$$\therefore \sqrt{1+k^2}=xy+\sqrt{(1+x^2)\ (1+y^2)}.$$

124. $1+x+\dfrac{4x^2}{3}+\text{etc.}$

125. $1+\dfrac{x}{2}+\dfrac{3x^2}{8}+\text{etc.}$

126. Assume $x^4+px^3+qx^2+rx+s=(x^2+ax+b)^2$.

Then equating coefficients of corresponding powers of x,

$$\begin{aligned}p&=2a,\\ q&=a^2+2b,\\ r&=2ab,\\ s&=b^2.\end{aligned}$$

Beginning with the last, $b=\sqrt{s}$; $\therefore r=2a\sqrt{s}$;

$$\therefore a=\frac{r}{2\sqrt{s}};\ \therefore q=\frac{r^2}{4s}+2\sqrt{s}.$$

$$p=2a=\frac{r}{\sqrt{s}};\ \therefore p^2=\frac{r^2}{s};\ \therefore p^2s=r^2.$$

128. c.

129. Let a, b, c, d be the digits.
Then the numbers will be

$$\left.\begin{array}{l} ar^3+br^2+cr+d \\ br^3+cr^2+dr+a \\ cr^3+dr^2+ar+b \\ dr^3+ar^2+br+c \end{array}\right\} =(a+b+c+d)(r^3+r^2+r^1).$$

If a number be taken which has more or less than four digits, the result will be similar; for example, if a, b, c, d, etc., . . . n be the digits, the sum of all the numbers will be
$(a+b+c+d+\text{etc.} \ . \ . \ . \ n)\ (r^{n-1}+r^{n-2}+\text{etc.} \ . \ . \ . \ r+1)$.

130. See answer to Ex. 126, and proceed in a similar way.

132. Multiply together on each side equality. Cancel terms which occur on both sides, and making $ad=bc$,

$$\frac{p}{q}=\frac{r}{s}.$$

133. $\dfrac{a+x}{b+x}=\dfrac{e}{f}$; $\therefore x=\dfrac{af-be}{e-f}$.

134. $a:b::c:d$, also $a:b::b:c$;

$$\therefore ad=bc;$$
$$\therefore a^2d=abc. \quad \text{But } ac=b^2;$$
$$\therefore a^2d=b^3;$$
$$\therefore a^3=d^3; \ \therefore a:d::a^3:b^3.$$

135. 3.

136. See Todhunter's *Algebra*, page 241.

137. 1 : 9.

138. (2) Let $a:a+m::a+n:a+p$.
Let $a+p$ be greatest, then $a+m>a$, and $a+n>a$;

$$\therefore a \text{ is least.}$$

We have then to show that $a+a+p>a+a+m+n$,

$$i.e., p>m+n.$$

Since $a:a+m::a+n:a+p$

$$a(a+p)=(a+m)\ (a+n);$$
$$\therefore a^2+ap=a^2+am+an+mn;$$
$$\therefore ap=am+an+mn;$$
$$\therefore p=m+n+\frac{mn}{a},$$

which is greater than $m+n$;

$$\therefore 2a+p>2a+m+n.$$

139. If $a : b :: c : d$, and $b = \sqrt{cd}$;

$$\therefore a : \sqrt{cd} :: c : d;$$
$$\therefore ad = c\sqrt{cd};$$
$$\therefore a^2d^2 = c^3d;$$
$$\therefore a^2d = c^3;$$
$$\therefore c^2 = \frac{a^2d}{c}. \quad \text{But } \frac{a}{c} = \frac{b}{d};$$
$$\therefore c^2 = \frac{a \cdot b \cdot d}{d} = ab;$$
$$\therefore c = \sqrt{ab}.$$

140. Let $A = mB^2$, $B^3 = nC^4$, $C^5 = pD^6$, $D^7 = qE^4$;

$$\therefore \frac{A}{B^2} \times \frac{B^3}{C^4} \times \frac{C^5}{D^6} \times \frac{D^7}{E^4} = mnpq;$$
$$\therefore \frac{A \cdot B \cdot C \cdot D}{E^4} = m \cdot n \cdot p \cdot q, \text{ which is constant.}$$

141. $a^3 + b^3$, and $a^3 - b^3$.

142. $9x^2 = 4y^2$.

143. $27 : 1$.

144. Let $\frac{x}{y}$ be the greater.

$$\text{Then } x > y;$$
$$\therefore x^2 > y^2;$$
$$\therefore bx^2 > by^2;$$
$$\therefore axy + bx^2 > axy + by^2;$$
$$\therefore x(ax + by) > y(ax + bx);$$
$$\therefore \frac{x}{y} > \frac{ax + by}{ay + bx};$$
$$\therefore \frac{ax + by}{ay + bx} < \frac{x}{y}.$$

In the same way it may be shown that $\frac{ax + by}{ay + bx} > \frac{y}{x}$.

146. The ratio is $\left(\frac{a + x}{a}\right)^n = \left(1 + \frac{x}{a}\right)^n$

$$= 1 + n\frac{x}{a} + n \cdot (n - 1)\frac{x^2}{a^2} + \text{etc.}$$

Now, if x is small compared with a, we may take $1 + \frac{nx}{a}$ to represent this series, *i.e.* $\frac{a + nx}{a}$;

$$\therefore a^n : (a + x)^n :: a : a + nx \text{ nearly.}$$

147. By addition and subtraction and multiplying by 2,

$$\frac{\sqrt{x}}{\sqrt{y}}=\frac{\sqrt[4]{x}}{\sqrt[4]{x-y}};\ \therefore\ \frac{x^2}{y^2}=\frac{x}{x-y};$$

$$\therefore\ \frac{y^2}{x^2}=\frac{x-y}{x}=1-\frac{y}{x};$$

$$\therefore\ \frac{y^2}{x^2}+\frac{y}{x}=1,$$

from which $\frac{y}{x}$ and $\frac{x}{y}$ may be found.

148. The ratio $=\dfrac{\left(a+\frac{x}{2}\right)-\left(a-\frac{x}{2}\right)}{a+\frac{x}{2}+a-\frac{x}{2}}$ nearly, $=\dfrac{x}{2a}$.

150. If $\frac{x}{y}+\frac{y}{x}$ is not >2, x^2+y^2 is not $>2xy$, or $(x-y)^2$ not >0; $\therefore$ it is either $=$ or <0. If less than 0, it is egative, which is impóssible, and it is only $=0$ when $=y$.

151. Number of permutations of 20 consonants taken o at a time $=20\times 19$. Then putting vowel in the ddle; as there are 5 vowels, the number of words $20\times 19\times 5=1900$.

152. There are 21 consonants and 5 vowels. We e two of the first and one of the second;

$$\therefore\ \text{number}=\frac{|21}{1.2}\times\frac{|5}{1}=\frac{|5\ |21}{1.2}.$$

54. $(10.9.8.7.6)^2$.

55. With 3 Protestants and 4 Catholics,

$$\text{number}=\frac{|20}{1.2.3}\times\frac{|15}{1.2.3.4}.$$

ith 4 Protestants and 3 Catholics,

$$\frac{|20}{1.2.3.4}\times\frac{|15}{1.2.3};$$

total number of selections

$$\left\{\frac{1}{1.2.3.4}+\frac{1}{1.2.3}\right\}+|15\left\{\frac{1}{1.2.3.4}+\frac{1}{1.2.3}\right\}$$

$$+|15)\times\frac{5}{1.2.3.4}=\frac{5(|20+|15)}{|4}.$$

157. $\frac{17\times 16}{1.2}\times\frac{5}{1.2}.\underline{|3}=2040.$

158. See page 41.

159. $\frac{12.11.10}{1.2.3}\times\frac{9.8.7.6}{1.2.3.4}\times\frac{5.4.3.2.1}{5.4.3.2.1}=\frac{\underline{|12}}{\underline{|5}\,\underline{|4}\,\underline{|3}}.$

160. $\frac{30.29.28.27.26}{1.2.3.4.5}.\quad\frac{25.24.23.22.21}{1.2.3.4.5}.$

161. See page 41.

162. See page 42.

163. See page 41.

165. $(1+n)^m-1.$

166. $\frac{1.3.5\ \ldots\ (4n-1)}{(1.3.5\ \ldots\ 2n-1)^2}.$

167. 255.

168. 120.

169. $n=11.$

170. 3796.

171. Let a' = first term, d = common difference;

$$\therefore a=(2a'+39d)\ 20,$$
$$b=(2a'+49d)\ 25,$$

from which $d=\frac{4b-5a}{1000}.$

172. See page 42.

173. Sum of $1+3+5+\text{etc.}=n^2$; $\therefore$ it would always form a square whose side was n.

174. If a^2, b^2, c^2 are in A. P.;

$$\therefore a^2+c^2=2b^2;$$
$$\therefore a^2+2ac+c^2=2b^2+2ac;$$
$$\therefore a^2+2ac+c^2+2ab+2bc=2b^2+2ac+2ab+2bc;$$
$$\therefore (a+c)\ (a+c+2b)=2(a+b)\ (b+c);$$
$$\therefore \frac{(a+2b+c)}{(a+b)\ (b+c)}=\frac{2}{a+c};$$
$$\therefore \frac{(a+b)+(b+c)}{(a+b)\ (b+c)}=\frac{2}{a+c};$$
$$\therefore \frac{1}{b+c}+\frac{1}{a+b}=\frac{2}{a+c}.$$

175. $n^2+n.\frac{2a-d}{d}=\frac{2S}{d}$, and a, d, and S are given.

176. $a=27$, $a+3d=18$; $\therefore d=-3$;

$$\therefore 117=\{54+(n-1)\ (1-3)\}\frac{n}{2}=(57-3n)\frac{n}{2};$$

$$\therefore 234 = 57n - 3n^2;$$
$$\therefore n^2 - 19n + 78\ (n-6)(n-13);$$
$$\therefore n = 6 \text{ or } 13.$$

177. Put $x=a$, $a+b$, $a+2b$, etc. $y=a^2+b$, a^2+ab+b, $a^2+2ab+b$, $a^2+3ab+b$, which are in A. P.

178. $n=1$.
179. See page 46.
180. $9n-36$.
181. See page 50.
182. $S=(a+l)\dfrac{n}{2}$.
183. See page 52.
184. $\frac{2}{3}\{1-(-\frac{1}{2})^n\}$. See page 49.
185. $\frac{3}{10}$.
186. Let x and y be the numbers.

The arithmetic mean, $\dfrac{x+y}{2}=1+a^2$.

The geometric mean, $\sqrt{xy}=1-a^2$.

Adding $x+y+2\sqrt{xy}=4$,

$$\sqrt{x}+\sqrt{y}=2.$$

Subtracting, $\sqrt{x}-\sqrt{y}=\pm 2a$;

$$\therefore \sqrt{x}=1\pm a;\ \therefore x=(1\pm a)^2.$$
$$\sqrt{y}=1\mp a;\quad y=(1\mp a)^2.$$

187. The series will be a, a^2, a^3, a being ratio, and 30 first term. Ratio $=\frac{2}{3}$ or $-\frac{5}{3}$.
188. $\frac{3}{2}$. See page 49.
189. See page 52.
190. First term $=1$.
um of first and second $=11$; $\therefore$ second term $=10$.
Sum of three terms $=111$; $\therefore$ third term $=100$;
$\therefore$ series is 1, 10, 100, 1000, etc.;
$$\therefore r=10.$$

91. See page 48.
92. See page 11.
3. $\dfrac{1+a}{1-a}$.
4. $a=1$, $r=\frac{1}{2}$.

195. Take each pair thus:

$$\frac{m+nx}{m-nx}=\frac{n+px}{n-px};\ \therefore\ \frac{m}{n}=\frac{n}{p};$$

$$\therefore\ mp=n^2;\ \therefore\ m,\ n,\ p \text{ are in G. P.}$$

Similarly, from the second and third, n, p, q are in G. P., etc.

196. Put x for the ratio.

The first series $=1+3x+5x^2+7x^3+\text{etc.}$

The second series $=1-3x+5x^2-7x^3+\text{etc.}$

To sum the first,

$$\text{Let } S=1+3x+5x^2+7x^3+\text{etc.};$$

$$\therefore\ Sx=x+3x^2+5x^3+\text{etc.}$$

$$\text{Subtracting, } S(1-x)=1+2x+2x^2+2x^3+\text{etc.}$$

$$=1+2\left(\frac{x}{1-x}\right)\text{ to infinity};$$

$$\therefore\ S(1-x)=\frac{1+x}{(1-x)};$$

$$\therefore\ S=\frac{1+x}{(1-x)^2}.$$

$$\text{The second, } S'=1-3x+5x^2-7x^3+9x^4+\text{etc.}$$

$$S'x=x-3x^2+5x^3-7x^4+\text{etc.}$$

$$\text{Adding, } S'(1+x)=1-2x+2x^2-2x^3+2x^4+\text{etc.}$$

$$=1-2(x-x^2+x^3-x^4+\text{etc.})$$

$$=1-2\left(\frac{x}{1+x}\right)$$

$$=\frac{1-x}{1+x};$$

$$\therefore\ S'=\frac{1-x}{(1+x)^2};$$

$$\therefore\ S:S'::\frac{1+x}{(1-x)^2}:\frac{1-x}{(1+x)^2},\text{ where } x=\tfrac{1}{2},$$

$$::\tfrac{3}{2}\div\tfrac{1}{4}:\tfrac{1}{2}\div\tfrac{9}{4}$$

$$::6:\tfrac{2}{9}$$

$$::27:1.$$

198. (a) Write the series in two, thus:

$$S=\frac{x}{10}+\frac{x}{10}+\frac{x}{10}+\text{etc.}-\left(\frac{1}{10}+\frac{2}{10}+\frac{3}{10}+\text{etc.}\right)$$

$$=20\times\frac{x}{10}-\frac{1}{10}(1+21)\ 10$$

$$=2x-21.$$

(β) $$\frac{(\sqrt{2})^{20}-1}{\sqrt{2}\sqrt{2}-1)}=\frac{2^{10}-1}{2-\sqrt{2}}.$$

199. (α) The series is double; substituting 1, 2, 3, etc. for n, we get $2+4$, $4+8$, $8+12$, etc.;

$$\therefore S=(2+4+8+\text{etc.})+(4+8+12+\text{etc.}).$$

$$\frac{2(2^n-1)}{2-1}+(4+4n)\frac{n}{2}$$

$$=2(2^n-1)+2n(n+1).$$

(β) Series is $(\sqrt{2}+2+2\sqrt{2}+\text{etc.})-1-2-3-4$

$$=\frac{\sqrt{2}\{(\sqrt{2^n})-1\}}{\sqrt{2}-1}-\frac{n(1+n)}{2}.$$

(γ) Series $=2+5+8=(2+3n-1)\frac{n}{2}=\frac{n(3n+1)}{2}.$

200. See page 53.

201. $x=1\frac{1}{2}$.

202. $x=2$ or $\frac{68}{21}$.

203. $x=1$.

204. See page 64.

205. See page 64.

206. $x=\frac{2}{3}$.

207. $x=1$.

208. $x=5$ or $5\frac{1}{2}$

209. $x=3$ or 5 or $4\pm\sqrt{10}$.

210. $x=4$.

211. $x=\frac{ab+ac+bc}{a+b+c}\pm\frac{\sqrt{a^2b^2+a^2c^2+b^2c^2-abc(a+b+c)}}{a+b+c}.$

212. $x=-\frac{11}{12}$.

213. $x=\frac{ab}{a+b}$.

214. $x=\frac{bh^3-fk^2}{ah^3-k^3(c+d)}$.

215. $x=\frac{1}{d\sqrt[3]{a^2+c}}-g$.

216. $x=\frac{2am}{m^2+1}$.

217. $x=\frac{a^2-b^2}{3a}$.

218. $x=3$.

219. $x = \frac{b^2 - 4a^2}{4a}$.

220. $x = \frac{b^2c^2 - a^2d^2}{2abcf}$.

221. $x = \pm\frac{a^2 + 1}{a^2 - 1}$.

222. $x = \pm 12$.

223. $x = \pm 5$.

224. $x = \pm\sqrt{\frac{2b + a}{2}}$.

225. $x = \frac{-a \pm b\sqrt{c^2 + 1}}{c}$.

226. $x = \pm 1 \cdot 609439$.

227. $x = \frac{1 \pm 2a \pm \sqrt{1 + 4a}}{2}$.

228. $x = -3 \cdot 6$.

229. $x = \frac{25}{16}$.

230. $x = \sqrt[4]{\frac{1 + \sqrt{2}}{2}}$.

231. $x = 3$ or 4. $y = 4$ or 3. See page 76.

232. $x = \frac{a}{\sqrt{a + b}}$. $y = \frac{b}{\sqrt{a + b}}$.

233. $x = a + b$. $y = \frac{b^3 - a^3}{ab}$.

234. $x = 2$ or $\frac{1}{2}$. $y = \frac{1}{2}$ or 2. See page 74.

235. See page 76.

236. $x = \frac{abc}{bc - ac + ab}$. $y = \frac{abc}{ac - bc - ab}$. See page 58.

237. See page 59.

238. $x = 3$. $y = \frac{1}{2}$. $z = 1$.

239. See page 59.

240. See page 75.

242. See page 57.

243. $z = \frac{167}{229}$.

244. $x = \frac{(a - b)^2}{a + b}$. $y = \frac{4ab}{a + b}$.

246. $x = \pm 5$. $y = \pm 3$.
247. $x = 2$ or 16. $y = 2$ or $\frac{1}{2}$.
248. $x = 0$ or $a - b$. $y = -b$ or $-a$.
249. $x = 4$ or $\frac{1}{4}$. $y = 2$ or 8.
250. $x = 4$ or $\frac{9}{\sqrt{12}}$. $y = 9$ or 12.
251. $x = 2$ or 3. $y = 3$ or 2.
252. $x = 25$. $y = 32$.
253. $x = \pm 2$. $y = \pm 3$. $z = \pm 4$.
254. $x = \{-\frac{1}{2} \pm \frac{1}{6}\sqrt{57}\}^8$. $y = \{-\frac{1}{2} \pm \frac{1}{6}\sqrt{57}\}^4$.
255. $x = 3$. $y = 1$.
256. $x = 3$. $y = 1$.
257. $x = 3$. $y = 1$.
258. $x = y = \sqrt[3]{a+b}$.
259. See page 86.
260. $x = 2$. $y = 1$.
261. $x = 5$. $y = 3$.
262. $x = 1$. $y = 0$.
263. $x = -6{\cdot}5$ or $\frac{53}{18}$. $y = \sqrt{30}$.
264. $x = \pm 2$. $y = \pm 2$.
265. $x = 3$. $y = 4$. $z = 5$.
266. $x = 2$. $y = 3$.
267. $x = 2$. $y = 3$.
268. $x = 5$. $y = 6$.
269. $x = \frac{1}{4}\{\sqrt{38 \pm 6\sqrt{5}} - 1 \pm \frac{3}{2}\sqrt{5}\}$.
270. $x = 3$ or 6. $y = 6$ or 3.
271. See page 89.
272. $x = 3$. $y = 2$.
273. $x = 3\{1 \pm \sqrt{3}\}$. $y = 3\left\{1 \pm \frac{1}{\sqrt{3}}\right\}$.
274. $x = 1$. $y = 2$. $z = 3$.
275. $x = 5$. $y = 6$. $z = 7$.
276. $x = \frac{\log b}{\log a}$.
277. $x = \frac{\log c}{\log a^m b^n}$.
278. $x = \frac{\log c^h d^k}{\log a^f b^g}$.
279. $x = 11$.

280. $x=1$.
281. $x=0$. $y=1$ or ± 2. $z=-3$, ± 2, or 1.
282. $x=\pm\frac{abc}{2}\left(\frac{1}{c^2}+\frac{1}{b^2}-\frac{1}{a^2}\right)$.
283. $x=a-b$. $y=b-c$. $z=c-a$.
284. $x=\frac{5}{2}$ or $\frac{1}{2}$. $y=\frac{1}{2}$ or $\frac{5}{2}$. $z=\frac{3}{2}$.
285. $x=5$. $y=3$. $z=2$.
286. $x=6$ or 2. $y=2$ or 6.
287. $x=\frac{3}{2}$ or $\frac{1}{2}$. $y=\frac{1}{2}$ or $\frac{3}{2}$. $z=3$.
288. $x=\frac{c}{2}(1\pm\sqrt{5})$. $y=\frac{c}{2}(-1\pm\sqrt{5})$.
289. $x=\frac{1}{2}(1+a\sqrt{3})$. $y=\frac{a}{2}\left(1-\frac{1}{\sqrt{3}}\right)^{\frac{1}{2}}$.
290. $x=30$, 15, or 27. $y=15$, 30, or 27.
291. $x=4$. $y=6$.
292. $x=\sqrt{\frac{a+b}{a-b}}$. $y=\sqrt{\frac{a-b}{a+b}}$.
293. $x=\pm 2\sqrt{2}$. $y=\pm 2\sqrt{3}$.
294. $x=2$ or -3. $y=3$ or -2.
295. $x=-\frac{52}{5}$. $y=\frac{13}{8}$.
296. $x=\frac{\sqrt[5]{10}}{2\sqrt{5}}(\sqrt{5}-1)$. $y=\frac{\sqrt[5]{10}}{2\sqrt{5}}(\sqrt{5}+1)$.
297. $x=4$. $y=\frac{1}{4}$.
298. $x=\pm 3$. $y=\pm 2$.
299. $x=8$ or -12. $y=2$ or $-\frac{4}{3}$.
300. £20.
301. £1200.
302. 39 yards and 29 yards.
303. 200 miles, speed = 4 miles per hour.
304. No.
305. $1000\left(\frac{5}{7}+\frac{5^2}{7^2}+\frac{5^3}{7^3}+\text{etc. } \textit{infin.}\right)=£25000$.
306. 6.
307. See page 72.
308. 32·425. 15·012.
309. 60 quarts.
310. 909 and 969.
312. 9. 20.
313. 15625.
314. 1·618. 2·616.

315. 2. 4. 8.
316. 3. 1.
317. 862.
318. 16.
319. 7. 4.
320. 8 cubic feet.
321. Euclid VI. 4.
322. The points are those at which the common tangents meet. If the two lines are on the same side of the circles, the point will be beyond the smaller circle. If on opposite sides, the point will be between the two circles.
323. Euclid VI. 18.
324. Join PQ.
Then OPQ is a right-angled triangle;
$\therefore PQ = OP\sqrt{5}$. This is the distance of Q from a fixed straight line.
$QQ = 2OP$, which is its distance from a point.
But $OP\sqrt{5}$ is $> 2OP$;
$\therefore$ distance from directrix is greater than from point;
$\therefore$ the locus is an ellipse.
325. Euclid XI. 8.
326. Euclid VI. 6.
327. If it is a proportional to the segments of base, the two triangles will be similar to each other. The two angles at ends of base will be equal to one right angle; $\therefore$ triangle is right-angled.
328. Euclid I. 43.
329. Through the point where the diagonal of the whole parallelogram meets EH, draw lines parallel to the sides AB and AD, and terminated by CB and CD produced; the proposition is then self-evident.
330. Euclid XI. 17.
331. Euclid VI. 2.
332. Draw perpendiculars from two angles, and the point of intersection will be required point.
333. Euclid XI. 11.
334. Join the points of contact, and join each to centre of sphere. Join centre of sphere with external point.
Then all the right-angled triangles thus formed have two sides equal each to each; $\therefore$ third sides are all equal.

335. The edges form tangents to the sphere, and are equal in pairs because they are drawn from the same external point.

336. Divide the rectangles into two triangles by diagonals, then corresponding triangles are similar. Hence rectangles are similar.

337. Euclid VI. 19.

338. See page 98.

339. Euclid XI. 4.

340. (1) See page 98. (2) Prove that all tangents are equal as in 334. Draw lines from points of contact of tangents with sphere perpendicular to line joining centre with extreme point. Then all these lines will be equal by I. 47. Show that any of them is less than radius of sphere. Hence the section is a small circle.

341. Euclid VI. 3.

342. Euclid VI. 5.

344. When it makes right angles with every line meeting it in that plane.

Euclid XI., Defs. 5 and 6. The inclination of a line drawn parallel to one which meets the other.

345. Euclid XI. 20.

346. Euclid XI., Defs. 14, 18, 21. A right cone is one in which the straight line joining the vertex with the middle point of the base is perpendicular to the plane in which the base lies. A right cylinder is one in which the axis is perpendicular to the base. If the hollow cone is not a right cone, then the tangents drawn from the vertex to the sphere would not be equal;
$\therefore$ the sphere could not fit in.

347. Let A be the point of contact of the two circles, B, C the points of contact of the common tangent with the two circles, O the point of intersection of the tangents. Then $OC = OA$, $OA = OB$; $\therefore OB = OC$.

348. Draw the diagonals. Then two of the sides of the figure formed are parallel to each diagonal, and the two triangles into which the figure is divided by the diagonal are each double the parallelogram on half the base and half the altitude.

349. Euclid VI. 31.

350. Euclid XI. 19.

351. See page 98.

352. Let ABC be the given circle, D the given point. Find the centre of the circle E. At E draw a line,

FEG, perpendicular to the plane of the circle. Then the centre of the sphere is in this line. Join DC. DC will be a chord of a great circle of the sphere. Bisect DC in H. Draw HK perpendicular to DC. Then the centre of the sphere is in DH produced; $\therefore$ the centre is the point of intersection of HK and FEG.

353. See page 109.

354. Euclid VI. 23.

355. The two bisectors include a right angle between them. AC : BC :: AD : BD.

356. See *Co-ordinate Geometry*, chap. x.

357. Euclid XI. 11.

358. The straight line is at right angles to any line which it meets in the plane. A plane may be drawn through this line and any other line in the plane, and then the given plane will be at right angles to this plane.

359. If the cone be cut down from the vertex to the base by a straight line, and unwrapped, the convex surface will be a sector of a circle; and as this may be regarded as equal to a number of triangles whose altitudes are the slant height of the cone, and whose total bases are equal in length to the arc of the sector,

$$\therefore \text{area} = \frac{\text{slant height} \times \text{circumference of base}}{2}.$$

Let a be the slant height, b the radix of base, c the altitude of the cone.

$$\text{The convex area} = ab\pi,$$

$$c^2 + b^2 = a^2;$$

$$\therefore a = \sqrt{b^2 + c^2};$$

$$\therefore \text{convex area} = \pi b\sqrt{b^2 + c^2}.$$

$$\text{Area of base} = (2b)^2 \frac{\pi}{4} = b^2\pi;$$

$$\therefore \text{total area} = \pi b(b + \sqrt{b^2 + c^2}).$$

360. Prism is three times the pyramid.

361. Divide the two polygons into triangles by corresponding lines. Then of two triangles, one in each figure, the two sides which are sides of the polygon are proportional, and the angles are equal; $\therefore$ corresponding lines which join opposite angles are proportional. Hence all the triangles are similar, each to each.

362. Each pair of lines will form a triangle with the

given line; and as the other line is parallel to all their bases, the segments will be proportional to the sides of the triangles formed, the bases of which are parallel to each other. See Euclid VI. 2.

363. Join the middle points of two of the sides. This line is parallel to the base, and equal to half of it. Let O, P, be the points, Q the point of intersection. Then if OP be parallel to BC, OPQ is similar to BCQ; but OP is half BC, ∴ OQ is half QC. Similarly, the proposition is true for the other lines.

364. See Euclid XI. 15.

365. See page 104.

366. See page 106.

367. See Euclid VI. 6.

368. Euclid VI. 19.

369. See page 98, note 2.

370. See Euclid XI. 21.

372. Bisect each of the vertical angles. The bisector will be the altitude. Prove angles at base are equal, then that the triangles formed by altitude, one side, and half base are similar; ∴ altitudes are as half bases; ∴ altitudes are proportional to bases.

373. Make a line equal to the perimeter of the given triangle, and divide the given perimeter similarly to it by Euclid VI. 10.

374. See Ex. 362. The proof depends upon the same principle.

375. From the point draw a perpendicular to the plane at O, then from the point two of the equal lines to the plane. Join the ends of these lines on the plane with O in the plane, and prove, by I. 47, that the distances from O are the same in all cases; ∴ the locus is a circle.

376. See page 98.

377. Let x = altitude of cone, a = diameter of base.

$$\text{Slant height} = \sqrt{x^2 + \left(\frac{a}{2}\right)^2} = \tfrac{1}{2}\sqrt{4x^2 + a^2};$$

$$\therefore \text{ area} = \tfrac{1}{2}\frac{\sqrt{4x^2 + a^2}\,.\,\pi a}{2},$$

$$\text{area of hemisphere} = \frac{a^2\pi}{2}.$$

The areas of the bases are equal;

$$\therefore \tfrac{1}{4}\sqrt{4x^2+a^2}\,\pi a = \frac{a^2\pi}{2};$$

$$\therefore \sqrt{4x^2+a^2} = 2a,$$

$$\text{and } 4x^2+a^2 = 4a^2,$$

$$4x^2 = 3a^2,$$

$$x = a\frac{\sqrt{3}}{2},$$

which is the altitude of an equilateral triangle whose base $= a$.

378. Euclid VI. 10.

379. Euclid VI. 25.

380. Euclid XI. 11.

381. From the given point draw a line perpendicular to the point of section; then show that this line is in the same plane with each of the other two. Or draw a line parallel to the line of intersection from the point, and show that it is perpendicular to the plane of the two perpendiculars.

382. See page 103.

383. See page 103.

384. See page 103.

385. Join the ends of each of the parallel sides, and the angles of one triangle are equal each to each to those of the other.

386. Divide the rectilinear figures into similar triangles by lines drawn from corresponding angles. Then prove by similar triangles.

387. Take AC as an independent line, and divide it similarly to ABC. Then from AC cut off AD, equal to the part which is proportional to AB.

388. See Euclid XI. 10.

389. 5 inches, by Euclid I. 47.

390. Let r = given radius.

Then circumference and altitude each $= 2r\pi$.

$$\text{Area of base} = (2r)^2\frac{\pi}{4} = \pi r^2.$$

$$\text{Altitude} = 2r\pi;$$

$$\therefore \text{vol.} = \frac{2\pi r \,.\, \pi r^2}{3} = \frac{2}{3}\pi^2 r^3.$$

$$\text{Area of curved surface} = \frac{2r\pi \times \text{slant height}}{2}.$$

If s = slant height,

$$s^2 = \text{alt.}^2 + \text{rad.}^2$$
$$= (2r\pi)^2 + r^2 = r^2(4\pi^2 + 1);$$
$$\therefore s = r\sqrt{4\pi^2 + 1};$$
$$\therefore \text{total surface} = \frac{2\pi r^2\sqrt{4\pi^2+1}}{2} + \pi r^2$$
$$= \pi r^2(\sqrt{4\pi^2 + 1} + 1).$$

391. 1 ounce nearly.
392. 36·64 feet.
393. $55\frac{1}{2}$ miles.
394. $693\frac{1}{2}$ inches.
395. 205·8 gallons.
396. 1·43.
397. 3·78 inches.
398. 1·598 feet.
399. The quantity will be that part of the volume of the sphere which is immersed = ·7580 of a pint.
400. $3\frac{1}{3}$ inches.
402. (1) $\theta = 45°$.
(2) $\theta = 60°$.

404. The formulæ for $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$, etc., are preferable for finding the angles, because we only require the logs of s, $s-a$, $s-b$, $s-c$ to find all the angles, but if we use the formula given in the preceding example, it is necessary to find the logs of the sides.

405. 90°.
406. See page 174.
407. See page 150.
408. See page 127.
409. $A = 180 - (B + C)$;

$$\therefore \sin A = \sin(180 - B - C) = \sin(B + C)$$
$$= \sin B \cos C + \sin C \cos B.$$

Also $\cos A = -\cos(B + C) = \sin B \sin C - \cos B \cos C$;

$$\therefore \sin^2 A = \sin^2 B \cos^2 C + \sin^2 C \cos^2 B$$
$$+ 2 \sin B \sin C \cos B \cos C$$
$$= \sin^2 B(1 - \sin^2 C) + \sin^2 C(1 - \sin^2 B)$$
$$+ 2 \sin B . \sin C . \cos B . \cos C$$
$$= \sin^2 B + \sin^2 C - 2 \sin B \sin C \{\sin B \sin C$$
$$- \cos C \cos B\}$$
$$= \sin^2 B + \sin^2 C - 2 \sin B . \sin C \cos A. \quad \text{(Q. E. D.)}$$

410. See page 122.
411. See page 133.
412. See page 148 *et seq.*
413. Let y = height of tree,
x = breadth of river.

$$\text{Then } \frac{y}{100+x}=\tan 30=\frac{1}{\sqrt{3}}.$$

$$\frac{y}{x}=\tan 60=\sqrt{3};$$

$$\therefore y=x\sqrt{3};$$

$$\therefore \frac{x\sqrt{3}}{100+x}=\frac{1}{\sqrt{3}};\ \therefore 3x=100+x;$$

$$\therefore x=50,\ y=50\sqrt{3}.$$

$\therefore$ tree is $50\sqrt{3}$ feet high, and river is 50 feet wide.

414. $$\frac{b}{c}=\frac{\sin B}{\sin C};\ \therefore \frac{\frac{b}{c}-1}{\frac{b}{c}+1}=\frac{\frac{\sin B}{\sin C}-1}{\frac{\sin B}{\sin C}+1};$$

$$\therefore \frac{b-c}{b+c}=\frac{\sin B-\sin C}{\sin B+\sin C}=\frac{\tan \frac{1}{2}(B-C)}{\tan \frac{1}{2}(B+C)};$$

$$\therefore b-c : b+c :: \tan \tfrac{1}{2}(B-C) : \tan \tfrac{1}{2}(B+C).$$

415. See page 175.

416. (1) Sin $x=0$ or $\sqrt{2}$. The first value gives 0° or 180°; the second is impossible.

(2) Sin $x=0$, cos $x=1$; $\therefore$ angle is 0 or 360°.

417. See page 172.

418. (1) $\text{Sin}\,(A+B)\sin(A-B)=\sin^2 A\cos^2 B$
$-\cos^2 A\sin^2 B$
$=\sin^2 A\,(1-\sin^2 B)-\sin^2 B\,(1-\sin^2 B)$
$=\sin^2 A-\sin^2 B.$

(2) $\text{Sin}^2 A+\cos^2 A=1.$
$2\sin A\cos A=\sin 2A.$

Subtracting, $(\sin A-\cos A)^2=1-\sin 2A;$

$$\therefore \sqrt{1-\sin 2A}=\sin A-\cos A.$$

419. See page 150.

420. Let A, c, and b be given. Then, as in No. 414 above,

$$\frac{b-c}{b+c}=\frac{\tan \frac{1}{2}(B-C)}{\tan \frac{1}{2}(B+C)}.$$

$$\text{But } \tan \tfrac{1}{2}(B+C)=\cot \tfrac{1}{2}A;$$

$$\therefore \tan \tfrac{1}{2}(B-C)=\frac{b-c}{b+c}\cot \tfrac{1}{2}A,$$

from which $B-C$ is easily found.

421. $\operatorname{Cos} A=\dfrac{b^2+c^2-a^2}{2bc}$. $\operatorname{Cos} B=\dfrac{a^2+c^2-b^2}{2ac}$.

$$\operatorname{Cos} C=\frac{a^2+b^2-c^2}{2ab};$$

$$\begin{aligned}\therefore\ & bc\cos A+ac\cos B+ab\cos C\\ &=\tfrac{1}{2}(b^2+c^2-a^2+a^2+c^2-b^2+a^2+b^2-c^2)\\ &=\tfrac{1}{2}(a^2+b^2+c^2).\end{aligned}$$

423. See page 135, and Exs. 414 and 420 above.

424. Let C be the foot of the tower.

$$\text{Then } \frac{100}{AC}=\tan 45=1;\ \therefore AC=100$$

$$\frac{100}{BC}=\tan 30=\frac{1}{\sqrt{3}};\ \therefore BC=100\sqrt{3}.$$

But BC is hypotenuse of a right-angled triangle ABC;

$$\begin{aligned}\therefore AB&=\sqrt{BC^2-AC^2}\\ &=\sqrt{30000-10000}\\ &=100\sqrt{2}.\end{aligned}$$

425. See page 112.

426. See page 170.

428. Let x be the height of the cliff above the top of the one from which the observations are made. $y=$ breadth of river.

$$\text{Then } \frac{200}{y}=\tan 30.\quad y=200\sqrt{3}, \text{ the width of river.}$$

$$\frac{x}{y}=\tan 45=1;$$

$$\therefore \frac{x}{200\sqrt{3}}=1;\ \therefore x=200\sqrt{3};$$

$$\therefore \text{height of cliff}=200(1+\sqrt{3}).$$

430. $\operatorname{Sin} 75^\circ=\dfrac{\sqrt{3}+1}{2\sqrt{2}}$. $\operatorname{Cos} 75^\circ=\dfrac{\sqrt{3}-1}{2\sqrt{2}}$.

$\operatorname{Sin} 75^\circ=\sin(30^\circ+45)$. $\operatorname{Cos} 75^\circ=\cos(30^\circ+45)$.

431. See page 174, Formula 211.

432. $\frac{1}{14}\sqrt{115}$. $\frac{4}{21}\sqrt{5}$. $\frac{1}{3}\sqrt{5}$.

434. See answer to Ex. 404.

435. Let ABCD be the base, E the vertex, F the middle point of AB, O the centre of base. Join AO, FO, EO. Let a = side of base, b the edge.

$$AE=b,\ AO=a\frac{\sqrt{2}}{2},\ OF=\frac{a}{2},$$

$$EO^2=EA^2-AO^2=\frac{2b^2-a^2}{2},$$

$$EF^2=EO^2+OF^2=\frac{4b^2-a^2}{4}.$$

The angle required is EFO, EFO is a right-angled triangle, and $\tan EFO=\frac{EO}{OF}=\sqrt{\frac{4b^2-2a^2}{4b^2-a^2}}$;

$\therefore$ the angle made by each face with the base is that whose tangent is $\sqrt{\frac{4b^2-2a^2}{4b^2-a^2}}$.

436. Put $b=a$ in last example, and reduce.

438. Let Δ be the area, and x the side not given.

$$\Delta^2=8(b+x-a)(a+x-b)(a+b-x)(a+b+x)$$
$$=8(2a^2b^2-a^2-b^2-x^2+2x^2(a^2+b^2));$$
$$\therefore x^2=\frac{\Delta^2-8(2a^2b^2-a^2-b^2)}{2(a^2+b^2)-1}.$$
$$x=\frac{\sqrt{\Delta^2-8(2a^2b^2-a^2-b^2)}}{\sqrt{2(a^2+b^2)-1}},$$

and the angles may be found from one of the formulæ, Nos. 223 to 227.

440. See page 171, Nos. 149, 150.

441. $\frac{\text{Sin A}}{a}=\frac{\sin\{180-(A+B)\}}{c}=\frac{\sin(A+B)}{c}$;

$$\therefore a=c.\frac{\sin A}{\sin(A+B)}.$$

$$\text{Similarly, } b=\frac{c\sin B}{\sin(A+B)},\ \frac{h}{a}=\sin A;$$

$$\therefore h=a.\sin A=c\frac{\sin^2 A}{\sin(A+B)}.$$

$$\text{Area}=\frac{hc}{2}=\frac{c^2\sin^2 A}{\sin(A+B)}.$$

442. See page 172.

443. In a triangle,

$$\sin C = \sin (A+B), \text{ and } \cos C = -\cos (A+B);$$

$$\therefore\ -\cos (A+B) \sin C = \sin (A+B) \cos C;$$

$$\therefore\ \sin C (\sin A \sin B - \cos A \cos B) = \cos C (\sin A \cos B + \cos A \sin B);$$

$$\therefore\ \sin A \sin B \sin C - \cos A \cos B \sin C = \sin A \cos B \cos C + \cos A \sin B \cos C;$$

$$\therefore\ \sin A \sin B \sin C = \sin A \cos B \cos C + \cos A \cos B \sin C + \cos A \sin B \cos C.$$

444. Area $= \sqrt{s(s-a)(s-b)(s-c)}$.

445. See page 171, No. 148.

446. Find A and B by formula used in Exs. 414, 420.

Then $c = a \,.\, \dfrac{\sin C}{\sin A}$.

447. $\frac{12}{13}$. $\frac{12}{15}$. $\frac{56}{65}$.

448. Sin $\theta = \frac{2}{5}\sqrt{5}$, cos $\theta = -\dfrac{1}{\sqrt{5}} = -\frac{1}{5}\sqrt{5}$.

449. In Ex. 443 it is shown that

$$\sin A . \sin B . \sin C = \sin A \cos B \cos C + \cos A . \sin B \cos C + \cos A \cos B . \sin C.$$

Dividing every term by cos A . cos B . cos C,

$$\frac{\sin A}{\cos A} \cdot \frac{\sin B}{\cos B} \cdot \frac{\sin C}{\cos C} = \frac{\sin A}{\cos A} + \frac{\sin B}{\cos B} + \frac{\sin C}{\cos C};$$

$$\therefore\ \tan A . \tan B . \tan C = \tan A + \tan B + \tan C.$$

450. Let A = place of observation, B the top of pole, C the point one-third up, D the base.

Then if x = height of pole, by right-angled triangles

$$AB = \sqrt{40^2 + x^2},\ AC = \tfrac{1}{3}\sqrt{120^2 + x},\ BC = \frac{2x}{3},\ CD = \frac{x}{3}.$$

If tan BAC $= \frac{1}{2}$, sin BAC $= \dfrac{1}{\sqrt{5}}$, and sin ABC $= \dfrac{40}{x}$;

$$\therefore\ \frac{40}{x} : \frac{1}{\sqrt{5}} :: \tfrac{1}{3}\sqrt{120^2 + x^2} : \frac{2x}{3};$$

$$\therefore\ \frac{80}{3} = \tfrac{1}{3} \cdot \frac{1}{\sqrt{5}} \cdot \sqrt{120^2 + x^2},$$

from which $x = 40\sqrt{11}$.

451. Let ABC be a triangle, AD the altitude.

$$\text{Area}=\tfrac{1}{2}\text{BC}\,.\,\text{AD}=\tfrac{1}{2}bc\sin \text{B}=\tfrac{1}{2}\,.\,\frac{a\,.\sin \text{B}}{\sin \text{A}}\,.\,a\,.\sin \text{C}.$$

$$=\tfrac{1}{2}a^2\,\frac{\sin \text{B}\,.\sin \text{C}}{\sin \text{A}}.$$

452. Sin A = sin (B + C) ; ∴ by substitution in the last example,

$$\text{area}=\tfrac{1}{2}a^2\,\frac{\sin \text{A}\,.\sin \text{C}}{\sin (\text{B}+\text{C})}.$$

453. See page 140, Ex. 1.

454. See page 140.

455. In any triangle,

$\sin \text{A}=\sin (\text{B}+\text{C})=\sin \text{B}\cos \text{C}+\cos \text{B}\sin \text{C}.$

Let B = C ; ∴ $\sin \text{A}=2\sin \text{C}\cos \text{B}$;

$$\therefore \cos \text{B}=\frac{\sin \text{A}}{2\sin \text{C}}.$$

456. $\dfrac{2c\sin \alpha\sin \beta}{\cos \alpha+\cos \beta}.$

457. $\text{Area}=\sqrt{s(s-a)\,(s-b)\,(s-c)}$

$$=\frac{abc}{s}\sqrt{\frac{s(s-a)}{bc}\,.\,\frac{s(s-b)}{ac}\,.\,\frac{s(s-c)}{ab}}$$

$$=\frac{2abc}{a+b+c}\sqrt{\frac{s(s-c)}{bc}}\,.\sqrt{\frac{s(s-b)}{ac}}\,.\sqrt{\frac{s(s-c)}{ab}}$$

$$=\frac{2abc}{a+b+c}\,.\cos \tfrac{1}{2}\text{A}\,.\cos \tfrac{1}{2}\text{B}\,.\cos \tfrac{1}{2}\text{C}.$$

458. $2b=c+a.$ Side of equilateral triangle $=\dfrac{a+b+c}{3}.$

Substitute in formulæ, area $=\sqrt{s(s-a)\,(s-b)\,(s-c)}$, and simplify, eliminating c.

459. $\dfrac{\text{Cot A}}{\cot \text{A}-\cot 3\text{A}}=\dfrac{\cos \text{A}\sin 3\text{A}}{\sin 2\text{A}},$

$$\text{and }\frac{\tan \text{A}}{\tan \text{A}-\tan 3\text{A}}=-\frac{\sin \text{A}\cos 3\text{A}}{\sin 2\text{A}}.$$

$$\therefore \frac{\cot \text{A}}{\cot \text{A}-\cot 3\text{A}}+\frac{\tan \text{A}}{\tan \text{A}-\tan 3\text{A}}$$

$$=\frac{\cos \text{A}.\sin 3\text{A}-\sin \text{A}\cos 3\text{A}}{\sin 2\text{A}}=\frac{\sin 2\text{A}}{\sin 2\text{A}}=1.$$

460. By substitution, we get $1-2\sin^2\theta+\sin\theta=1$;

$$\therefore 2\sin^2\theta=\sin\theta;$$

$\therefore \sin\theta=0$ or $\frac{1}{2}$; $\therefore \theta=0$, or 180, or 360. $2n\pi$, or 30, or 150, etc.

461. Reduce each side of equation to sines, expand and simplify, which gives $\sin x$. Find $\sin 2x$ from this result.

462. $\pm\dfrac{2n(p^2-1)\pm(n^2-1)}{(n^2-1)(p^2-1)\mp 4pn}$.

463. Find $\sin 15°$ from $30°$ and $45°$, and then apply formula for half the angle.

464. See Ex. 409. Substitute $\cos^2 A-1=\sin^2 A$, etc.

465. $$\text{Tan } A+\frac{1}{\tan A}=\frac{\tan^2 A+1}{\tan A}=\frac{\dfrac{\sin^2 A}{\cos^2 A}+1}{\dfrac{\sin A}{\cos A}}.$$

$$=\frac{1}{\cos^2 A}\times\frac{\cos A}{\sin A}=\frac{1}{\sin A\cos A}$$

$$=\frac{2}{2\sin A\cos A}=\frac{2}{\sin 2A}.$$

466. Write $$1+\sin 2\theta=\frac{1+\tan\theta}{1-\tan\theta}=\frac{\sin\theta+\cos\theta}{\cos\theta-\sin\theta}$$

$$=\frac{\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta}{\cos^2\theta-\sin^2\theta};$$

$$\therefore 1+\sin 2\theta=\frac{1+\sin 2\theta}{\cos^2\theta-\sin^2\theta};\ \therefore \cos^2\theta-\sin^2\theta=1;$$

$$\therefore 1-\sin^2\theta-\sin^2\theta=1;$$

$$\therefore \sin^2\theta=0,\ \sin\theta=0,\ \cos\theta=1;\ \therefore \theta=0.$$

467. ·914.

468. Reduce the right-hand side to terms of cosine. Substitute for $\tan 2A$ and $\tan A$, and the equality will be evident.

470. Find the value of $\sin A$, then of $\cos A$, and then of the second quantity.

471. $(\text{Sin } A+\cos A)(\tan A+\cot A)$

$$=(\sin A+\cos A)\left(\frac{\sin A}{\cos A}+\frac{\cos A}{\sin A}\right)$$

$$=(\sin A+\cos A)\,\frac{(\sin^2 A+\cos^2 A)}{\sin A\cos A}=\frac{\sin A+\cos A}{\sin A\cos A}$$

$$=\frac{1}{\cos A}+\frac{1}{\sin A}=\sec A+\operatorname{cosec} A.$$

472. $40°$ and $50°$.

473. Tan $2A=\frac{2\tan A}{1-\tan^2 A}$; $\therefore \tan A=\frac{2\tan\frac{1}{2}A}{1-\tan^2\frac{1}{2}A}$.

Let $x=\tan\frac{1}{2}A$;

$$\therefore \tan A=\frac{2x}{1-x^2};\ \therefore \tan A-x^2\tan A=2x;$$

$$\therefore x^2-\frac{2}{\tan A}x=1;\ \therefore x^2-\frac{2}{\tan A}x+\frac{1}{\tan^2 A}=\frac{1+\tan^2 A}{\tan^2 A};$$

$$\therefore x=\frac{1\pm\sqrt{1+\tan^2 A}}{\tan A}.$$

474. Write $\frac{a^2+b^2}{a^2-b^2}=\frac{\sin(A+B)}{\sin(A-B)}$. Take sum and difference of numerator and denominator of each side to make new fractions.

475. W.N.W.

476. $3{\cdot}327$.

477. See page 172. $\frac{4\tan A-\tan^3 A}{1-6\tan A+5\tan^4 A}$.

478. $7{\cdot}7$.

479. $\sqrt{4-2\sqrt{2}}$, and $\sqrt{4+2\sqrt{2}}$.

481. $A=45°$.

482. Transpose $\sin^6 A$, take out the factor $\sin^2 A+\cos^2 A=1$ from left side, and express the whole quantities in sines or cosines.

483. If C had been a right angle, $\tan A=\cot B=\frac{1}{\tan B}$. If C is greater than a right angle, A+B is less than 90; and if A is constant, B will decrease as the angle decreases.

485. See Ex. 452.

486. See Ex. 363.

487. Let ABC be the triangle, O the centre of the inscribed circle, and D, E, F the points at which the circle touches the triangle. Let r be the radius.

Then area of triangle = area of AOB + BOC + COA

$$\therefore \sqrt{s(s-a)(s-b)(s-c)} = \text{area of AOB} + \text{BOC} + \text{COA}$$

$$= \frac{rc}{2} + \frac{ra}{2} + \frac{rb}{2};$$

$$\therefore r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

488. Let DEF be the middle points of the sides of the triangle. Then angle $A = \frac{1}{2}BOC$;

$$\therefore \sin A = \sin\frac{BOC}{2} = \sin BOE = \frac{BE}{BO} = \frac{a2}{2R}, \text{ R being radius;}$$

$$\therefore \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)} = \frac{a^2}{2R};$$

$$\therefore R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}.$$

491. This is derived directly from Formulæ 182, 183, page 172.

492. $x^3 + \frac{1}{x^3}$.

493. $\sqrt{\frac{c-1}{c+1}}$.

495. Expand sin (A+B+C), and simplify by cancelling out terms common to the several parts of the fraction. See Formula 146, page 171.

497. $3 \cdot \frac{1-\tan A}{1+\tan A} = \frac{1+\tan A}{1-\tan A}$, from which $\tan A = 2 \pm \sqrt{3}$.

QUESTIONS FOR 1879.

First B.A. and First B.Sc. Examinations.

ARITHMETIC AND ALGEBRA.

Examiners—DR. JOHN HOPKINSON, M.A., F.R.S.; REV. PROFESSOR TOWNSEND, M.A., F.R.S.

1. Multiply together $1\cdot3\dot{4}$ and $2\cdot5\dot{6}\dot{7}$, and extract the square root of the result to seven significant figures.

2. At what rate of compound interest will a given sum be increased eleven-fold in 100 years?
[$\log 11 = 1\cdot0413927$, $\log 1\cdot1266 = 0\cdot0517697$, $\log 1\cdot1267 = 0\cdot0518083$.]

3. A man of 25 years of age can insure his life for £1000 by paying an annual premium of £18. Taking interest at £5 per cent. per annum, what will be an equitable composition for an annual subscription of £3?

4. Ten English labourers can do as much excavating in six days as nine French labourers in seven days. A Frenchman receives one franc per cubic metre. How many pence must an Englishman receive per cubic yard, that his daily earnings may be five per cent. more than a Frenchman's?
[A metre may be taken $= 39\frac{3}{8}$ inches, and a franc $=$ 10d.]

5. The sum of the squares of two numbers is 1105, and their product is 552 times their difference. What are they?

6. There are m white men and n black men, n being greater than m. Find the number of ways in which each white man may have a black servant. If a white man may have any number of servants, in how many ways may every black man have a master?

7. Solve the equation $\dfrac{x-a}{b}+\dfrac{x-b}{a}=\dfrac{b}{x-a}+\dfrac{a}{x-b}$.

8. Under what conditions will x^3+ax^2+bx+c be divisible by x^2+px+q?

9. Sum the series $1+2x+3x^2+4x^3+\ .\ .\ .\ nx^{n-1}$.

10. A sum of £1000, bearing interest at 5 per cent., is to be paid off in three annual instalments, the payments, including the interest due, to be the same each year, and the first payment to be due at the end of the first year. What must the yearly payments be?

11. Define a logarithm. Prove the rules of multiplication and division of numbers by the aid of logarithms. Wherein lies the convenience of our tables being calculated to base 10?

12. What are oranges a gross, when fifty more for a sovereign lowers the price twopence a score?

GEOMETRY AND TRIGONOMETRY.

Examiners—DR. JOHN HOPKINSON, M.A., F.R.S.; REV. PROFESSOR TOWNSEND, M.A., F.R.S.

1. Two rectilinear triangles being supposed equiangular, show that their three pairs of homologous sides, opposite to their three pairs of equivalent angles, are proportional.

2. Four segments of a straight line being supposed proportional, show that every four rectilinear figures, similar in form and similarly described on them, are also proportional.

3. Three planes, not having a common line of intersection, being supposed such that two of their three lines of intersection are parallel, show that the whole three are parallel.

4. A plane being supposed to intersect a sphere, show that the intersection of the two is a circle whose centre is the foot of the perpendicular upon the plane from the centre of the sphere.

5. Three cylinders being supposed to pass through the three vertices, and to have for axes the three opposite sides of a rectilinear triangle, show that their three curved surfaces are equal in area.

6. A cone and a hemisphere being supposed to have equal bases and altitudes, determine the ratios of (*a*) their convex surfaces; (*b*) their entire volumes.

7. A rectilinear triangle being supposed circumscribed

by a circle, show that the ratio of any side to the diameter of the circle is equal to the sine of the opposite angle.

8. By aid of the preceding or otherwise, express the diameter of the circle circumscribing a rectilinear triangle in terms of the three sides, a, b, c, of the triangle.

9. Given, for three trigonometrical angles, α, β, γ, that $\tan\alpha=a$, $\tan\beta=b$, $\tan\gamma=c$. Find the value of $\tan(\alpha+\beta+\gamma)$ in terms of a, b, c.

10. Given of a rectilinear triangle the base c, and the two angles A and B. Find, in terms of them, the two sides a and b, the altitude h, and the area Δ.

11. The co-ordinates (rectangular) of the points P, Q, R, in a plane being respectively (3, 4), (5, 6), (7, 8), determine by any method the area of the triangle PQR.

12. The equations, in rectangular co-ordinates, of a circle in a plane, and of a right line through its centre, being respectively $x^2+y^2=r^2$, and $x+y=0$, find, by any method, those of the two tangents to the circle which are parallel to the line.

SOLUTIONS OF QUESTIONS FOR 1879.

ARITHMETIC AND ALGEBRA.

1. $1{\cdot}3\dot{4}=\dfrac{121}{90}$,

$$2{\cdot}5\dot{6}\dot{7}=\frac{2542}{990},$$

$$1{\cdot}3\dot{4}\times 2{\cdot}5\dot{6}\dot{7}=\frac{307582}{990\times 90}$$

$$=3{\cdot}45\dot{2}6987654\dot{3}.$$

$$\sqrt{1{\cdot}3\dot{4}\times 2{\cdot}5\dot{6}\dot{7}}=\sqrt{3{\cdot}45\dot{2}6987654\dot{3}09}$$

$$=1{\cdot}8579824.$$

2. We have

$$M=PR^n.$$

$$\text{Here } M=11P,$$

$$11P=PR^n;$$

$$\therefore R^n=11;$$

$$\therefore \log R\times 100=\log 11,$$

$$\log R=\frac{\log 11}{100},$$

$$\log R=\cdot010413927.$$

This does not correspond to log given; manipulate by adding, dividing, subtracting, etc.

Here by chance add log 1·1 to log R.

$$\begin{array}{r} \cdot 0413927 \\ \cdot 010413927 \\ \hline \cdot 051806627 \end{array}$$

$$\text{Log R} + \log 1{\cdot}1 = {\cdot}051806627.$$

$$\begin{array}{rll} \text{Log of } 1{\cdot}1266 = & {\cdot}0517697 & \quad 38{\cdot}6 = {\cdot}1 \\ 7 = & {\cdot}0518083 & \quad 3{\cdot}86 = {\cdot}01 \\ \hline \end{array}$$

$$\begin{array}{rl} \text{Difference for 1 is} & {\cdot}0000386 \\ \text{Log unknown} = & {\cdot}051806627 \\ \text{Log } 1{\cdot}1266 = & {\cdot}051769700 \\ \hline & \ldots\ldots 36927 \end{array}$$

$$\begin{array}{ll} \text{Then R} \times 1{\cdot}1 = 1{\cdot}126697 & \quad 386)36927(97 \\ \text{R} = 1{\cdot}011518 & \quad 2187 \end{array}$$

whence r is calculated.

3. When a man insures his life, he may be looked upon as paying so much per annum, and investing it at so much per cent.

Let x years be the expectation of life. Then formula in such cases is:

$$\text{Whole amount} = \text{A} \,.\, \frac{\text{R}^n - 1}{\text{R} - 1}.$$

$$\text{In this case } 1000 = 18\frac{\text{R}^x - 1}{\text{R} - 1};$$

$$\therefore \text{R}^x = \frac{34}{9}.$$

Again, present value of annuity of £3 for x years

$$= \text{A}\frac{\left\{1 - \frac{1}{\text{R}^x}\right\}}{\text{R} - 1} = \frac{3\left\{1 - \frac{9}{34}\right\}}{\frac{1}{20}}$$

$$= £\frac{3 \times 25 \times 20}{34} = £44\frac{2}{17}.$$

4. As 60 Englishmen do as much work as 63 Frenchmen, the additional value of the labour is 5 per cent. in favour of the Englishman.

Hence the question comes to this, that if the work is 10d. a cubic metre, how much will it be a cubic yard?

$$1 \text{ cubic metre} = 39\tfrac{3}{8} \times 39\tfrac{3}{8} \times 39\tfrac{3}{8} \text{ inches.}$$
$$1 \text{ cubic yard} = 36 \times 36 \times 36 \text{ inches;}$$

$\therefore$ by proportion,

$$39\tfrac{3}{8} \times 39\tfrac{3}{8} \times 39\tfrac{3}{8} : 36^3 :: 10\text{d.} : 7\tfrac{3}{4}\text{d. nearly.}$$

5. Let x and y be the two numbers.

Then $x^2+y^2=1105$,
$xy=552\ (x-y)$.

Put $a=552$, then $1105=2a+1$;

$\therefore$ the equations become

$$\left.\begin{aligned} x^2+y^2&=2a+1, & (1)\\ xy&=a(x-y). & (2)\end{aligned}\right\}$$

By (2), $x^2y^2=a^2(x^2-2xy+y^2)$.

Substituting, from (1), $x^2y^2=a^2(2a+1-2xy)$
$=2a^3+a^2-2a^2xy$;

$\therefore x^2y^2+2a^2xy=2a^3+a^2$;

$\therefore x^2y^2+2a^2xy+a^4=a^4+2a^3+a^2$
$=a^2(a+1)^2$;

$\therefore xy+a^2=\pm a(a+1)=a^2+a$;

$\therefore xy=a$;

$\therefore$ by (2), $x-y=1$.

And (1) becomes $x^2+y^2+2xy=2a+2a+1$
$=4a+1$;

$\therefore x+y=\sqrt{4\times 552+1}$
$=\sqrt{2109}=\pm 47$;

$\therefore x+y=\pm 47$,
$x-y=1$;

Adding, $2x=48$; $\therefore x=24$ or 23.

Subtracting, $y=23$ or 24.

6. See page 41.

7. $\dfrac{x-a}{b}+\dfrac{x-b}{a}=\dfrac{b}{x-a}+\dfrac{a}{x-b}$;

$\therefore a(x-a)^2(x-b)+b(x-b)^2(x-a)$
$=ab^2(x-b)+a^2b(x-a)$;

$\therefore (x-a)\ (x-b)\ \{a(x-a)+b(x-b)\}$
$=ab\{a(x-a)+b(x-b)\}$;

$\therefore (x-a)\ (x-b)=ab$,
$x^2-x(a+b)+ab=ab$,
$x=0$, and $x=a+b$.

Again, $a(x-a)+b(x-b)=0$;

$$\therefore ax-a^2+bx-b^2=0,$$
$$x(a+b)=a^2+b^2;$$
$$\therefore x=\frac{a^2+b^2}{a+b}.$$

8. Let x^2+px+q be contained $(x+m)$ times in

$$x^3+ax^2+bx+c;$$
$$\therefore (x^2+px+q)\,(x+m)=x^3+ax^2+bx+c;$$
$$\therefore x^3+x^2(p+m)+x(q+mp)+mq=x^3+ax^2+bx+c.$$

Equating corresponding coefficients,

$$p+m=a, \quad . \quad . \quad . \quad (1)$$
$$q+mp=b, \quad . \quad . \quad . \quad (2)$$
$$mq=c; \quad . \quad . \quad . \quad (3)$$
$$\therefore \text{from } (1),\ m=a-p;$$
$$\text{from } (2),\ q+p(a-p)=b;$$
$$\text{from } (3),\ q(a-p)=c;$$
$$\therefore q=\frac{c}{a-p};$$

$\therefore$ the condition is that

$$\frac{c}{a-p}+p(a-p)=b,$$
$$\text{or } c+p(a-p)^2-b(a-p)=0.$$

9. See page 53, Ex. 13.

10. £1000 in 3 years will amount to £1157·625. We have to find the amount of three annual payments which with interest will amount to this sum.

Let $x=$ the amount.

$$\text{Then } M=x\,.\,\frac{R^n-1}{R-1},$$
$$1157{\cdot}625=x\,.\,\frac{1{\cdot}05^3-1}{{\cdot}05};$$
$$\therefore x=\frac{1157{\cdot}625\times{\cdot}05}{1{\cdot}05^3-1}$$
$$=\frac{1157{\cdot}625\times{\cdot}05}{{\cdot}157625}$$
$$=£303{\cdot}7675.$$

11. See pages 17 to 24.

12. By question, 2d. a score less $=\frac{36}{5}$ of 2d. $=7\frac{2}{5}$d. $=\frac{6}{5}$s. per gross less.

Let $x=$ original price per gross in shillings.

Then $x-\frac{6}{5}$ = lowered price per gross in shillings.

Then $\frac{144\times 20}{x}$ = number for a sovereign at original price,

and $\frac{144\times 20}{x-\frac{6}{5}}$ = number for a sovereign at lowered price;

$$\text{therefore } \frac{144\times 20}{x}=\frac{144\times 20}{x-\frac{6}{5}}-50,$$

$$\frac{288}{x}=\frac{288}{x-\frac{6}{5}}-5,$$

$$288\left(x-\frac{6}{5}\right)=288x-5x\left(x-\frac{6}{5}\right)$$

$$-\frac{1728}{5}=-5x^2+6x,$$

$$x^2-\frac{6}{5}x=\frac{1728}{25},$$

$$x^2-\frac{6}{5}x+\frac{9}{25}=\frac{1737}{25},$$

$$x-\frac{3}{5}=\frac{\sqrt{1737}}{5}.$$

Now nearest root is 42;

$$\therefore\ x=\frac{45}{5} \text{ nearly} = 9\text{s. per gross.}$$

GEOMETRY AND TRIGONOMETRY.

1. See Euclid VI. 4.
2. Euclid VI. 22.
3. See figure to Euclid XI. 9.

Let HF, KF, BK be the planes.

Let AB, CD be parallel.

∠HGK = ∠BFD (XI. 10).

Similarly, ∠GHK = ∠FBD,

and ∠HKG = ∠BDF.

But AB is parallel to CD; ∴ HK = BD,

and triangle HKC = triangle BFD.

Join G with M, the middle point of HK; and F with N, the middle point of BD.

Then GM, FN are equal and parallel;
$\therefore$ MN is parallel to AB or CD (Euclid I. 33).
But MN is parallel to GF;
$\therefore$ AB, CD, GF are all parallel. (Q.E.D.)

4. See page 98.

5. Let ABC be any triangle, AE, BD, CF the perpendiculars from the angles on the opposite sides.

$$\text{Then } \frac{BD}{BC} = \sin C.$$

$$\text{Also } \frac{AE}{AC} = \sin C;$$

$$\therefore \frac{BD}{BC} = \frac{AE}{AC} = \frac{BD}{AB}.$$

Now, the cylinder having BC for its axis and passing through A, will have its convex surface $= 2AE . \pi . BC$.

And the cylinder having AC for its axis and passing through B, will have its convex area $= 2BD . \pi . AC$.

We have shown that

$$\frac{BD}{BC} = \frac{AE}{AC};$$

$$\therefore BD . AC = BC . AE;$$

$$\therefore 2BD . AC . \pi = 2BC . AE\pi;$$

$\therefore$ the areas of the convex surfaces of the two cylinders above are equal.

Similarly, the cylinder having AB for its axis and passing through C, will have its convex area equal to that of the one having AC for its axis and passing through B;

$\therefore$ all three cylinders have their convex areas equal.

6. See page 169, formulæ 101 and 102.

$$\text{Area of hemisphere} = \frac{C \times D}{2},$$

$$\text{area of cone} = \frac{C \times S}{2};$$

$$\therefore \text{area of hemisphere : area of cone} :: \frac{CD}{2} : \frac{CS}{2} :: D : S.$$

Now, S is the slant height, and as the height of cone = radius of base, slant height $= r\sqrt{2}$;

$$\therefore S = r\sqrt{2} = \frac{D}{2} . \sqrt{2} = \frac{D}{\sqrt{2}};$$

$$\therefore \text{D} : \text{S} :: \text{D} : \frac{\text{D}}{\sqrt{2}};$$

$\therefore$ area of hemisphere : area of cone :: $1 : \frac{1}{\sqrt{2}} :: \sqrt{2} : 1$.

Again, let r = radius of hemisphere and height of cone;

$$\therefore \text{ solidity of hemisphere} = \text{D}^3 \times \frac{\pi}{6} \times \frac{1}{2}$$

$$= (2r)^3 \times \frac{\pi}{6} \times \frac{1}{2}$$

$$= \frac{8r^3\pi}{12} = \frac{2r^3\pi}{3}.$$

$$\text{Solidity of cone} = \text{A} \times \frac{\text{H}}{3} \text{ (A being area of base)}$$

$$= (2r)^2 \times \frac{\pi}{4} \times \frac{r}{3}$$

$$= \frac{4r^2 . \pi . r}{4 . 3} = \frac{4r^3\pi}{12} = \frac{r^3\pi}{3};$$

$\therefore$ solidity of hemisphere : solidity of cone :: 2 : 1.

7. Let ABD be a triangle circumscribed by a circle. Let E be the centre of the circle.

Join AE, and produce AE to meet the circumference in C. Join BC.

Then ABC is a right-angled triangle.

From B draw BD perpendicular to AC.

$$\text{Then } \sin \text{C} = \frac{\text{BD}}{\text{BC}}.$$

$$\text{But by Euclid VI. 8, } \frac{\text{BD}}{\text{BC}} = \frac{\text{AB}}{\text{AC}};$$

$$\therefore \frac{\text{AB}}{\text{AC}} = \sin \text{C}.$$

But angle ACB or C = angle ADB (Euclid III. 21);

$$\therefore \frac{\text{AB}}{\text{AC}} = \sin \text{ADB}.$$

Similarly for other ratios.

8. Let c be the side opposite the angle ADB in the figure to the last example.

Let D be the diameter of the circle.

$$\text{Then } \frac{c}{\text{D}} = \sin \text{ADB}.$$

$$\text{But } \sin \text{ADB} = \frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)}$$

by formula 195, page 173;

$$\therefore \frac{c}{\text{D}} = \frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)};$$

$$\therefore \frac{1}{\text{D}} = \frac{2}{abc}\sqrt{s(s-a)(s-b)(s-c)};$$

$$\therefore \text{D} = \frac{abc}{2\sqrt{s(s-a)(s-b)(s-c)}}.$$

9. Tan $(\alpha+\beta+\gamma)$

$$= \frac{\tan\alpha + \tan\beta + \tan\gamma - \tan\alpha . \tan\beta . \tan\gamma}{1-(\tan\alpha\tan\beta + \tan\alpha\tan\gamma + \tan\beta\tan\gamma)}$$

$$= \frac{a+b+c-abc}{1-ab-ac-bc}. \quad \text{(See page 171.)}$$

10. See Example 441, page 218, and its solution, page 247.

11. Area $=0$. The three points are in a straight line, of which the equation is $y-x=1$.

12. The given lines will cut the circle in the points $x = \pm\frac{r\sqrt{2}}{2}, y = \pm\frac{r\sqrt{2}}{2}$.

And since the tangent is represented by

$$y = mx + c\sqrt{1+m^2},$$

the two tangents will be $y = -x \pm r\sqrt{2}$;

$$\therefore y+x = r\sqrt{2},$$

and $y+x = -r\sqrt{2}$ will be the equations to the required tangents.

N.B.—The last two questions will be worked fully in Part II.

STEWART'S MATHEMATICAL SERIES.

THIS, the most complete Mathematical Series ever issued, has occupied several years in preparation, and has been written by a number of the more eminent mathematical professors and teachers—the ripest mathematical scholarship of the age having been laid under contribution. The Engravings, numbering many hundreds, are executed (in the more expensive of the works at least) in the very highest style; on the printing and general get-up no expense has been spared, and the prices have been fixed as low as is compatible with their being at all remunerative.

	s.	*d.*
1. COMPLETE TREATISE ON PRACTICAL GEOMETRY AND MENSURATION, ...	5	0
2. KEY TO THE COMPLETE TREATISE ON PRACTICAL GEOMETRY AND MENSURATION, with full demonstrations of the rules and solutions of all the more difficult exercises, for use by the Teacher and the more advanced Mathematical Student, ...	6	0
3. COMPLETE PRACTICAL TREATISE ON THE NATURE AND USE OF LOGARITHMS, and on Plane Trigonometry, with Logarithmic and Trigonometrical Tables, . .	5	0
4. KEY TO PRACTICAL TREATISE ON THE NATURE AND USE OF LOGARITHMS, and on Plane Trigonometry, . .	3	0
5. ADVANCED TREATISE ON ALGEBRA. (*In the Press*), .	5	0
6. COMPLETE KEY TO ADVANCED TREATISE ON ALGEBRA, .	6	0
7. ELEMENTARY PRACTICAL GEOMETRY AND MENSURATION, .	2	6
8. KEY TO PRACTICAL GEOMETRY AND MENSURATION, .	2	6
9. COMPLETE KEY TO ELEMENTARY ALGEBRA, . .	2	6
10. ELEMENTARY LOGARITHMS AND TRIGONOMETRY, with Logarithmic and Trigonometrical Tables, . .	2	6
11. KEY TO ELEMENTARY LOGARITHMS AND TRIGONOMETRY, .	2	6
12. ELEMENTARY ALGEBRA, ...	2	0
13. COMPLETE KEY TO ELEMENTARY ALGEBRA, . .	2	6
14. TABLES OF LOGARITHMS, ...	1	0
15. EUCLID, FIRST FOUR BOOKS, WITH EXERCISES, . .	1	0
16. CONIC SECTIONS, ...	1	0
17. TRIGONOMETRY AND MENSURATION, . . .	1	6
18. MECHANICS (STATICS), ...	1	0
19. ALGEBRA (ELEMENTS OF), ...	1	0
20. RIDERS, DEDUCTIONS, AND EXERCISES ON THE FIRST FOUR BOOKS OF EUCLID, ...	3	6